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A NEW METHOD OF TASK SPECIFICATION FOR SPHERICAL MECHANISM DESIGN

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ABSTRACT

In this paper we present a novel method for motion generation task specification for spherical mechanisms. This is accomplished with a new methodology for determining the optimal design sphere and the orientations on this design sphere for a finite set of desired spatial positions. In addition, we include a modification to the method which enables the designer to require that one of the n desired spatial positions be exactly preserved. The result is that designers can now specify spherical mechanism motion generation tasks without having to introduce into the design space an artificial design sphere. They are now free to work in unconstrained three-dimensional space. The application of this new task specification technique is discussed in a design case study.

INTRODUCTION

Spherical mechanisms are linkages which generate motion on concentric spheres and are the simplest mechanisms which provide spatial movement. In contrast, planar mechanisms generate two-dimensional motion. For this reason their design is compatible with using conventional drafting tools while the synthesis of spherical mechanisms is three-dimensional and is not well suited for drafting techniques. It is essential that the spherical mechanism designer be able visualize the entire problem in three-dimensions. Computer graphics can be an effective tool for providing this necessary visualization of the problem to the designer. Efforts have been made to create computer graphics based software packages for spherical four-bar mechanism design.

*S*PHINX was the first spherical mechanism computer-aided design(CAD) program written by Larochelle et al 1993 for use on Silicon Graphics workstations. *S*PHINX begins by displaying a **design sphere**. The design sphere defines the surface in space upon which the workpiece is to be moved. The relative displacements between the positions on the design sphere are purely rotational and are called **orientations**. Orientations are defined by their longitude, latitude, and roll angles(Larochelle and McCarthy 1995). In *S*PHINX orientations are displayed to the designer as coordinate frames on the surface of the design sphere, see Fig. 1. The current version of *S*PHINX has modules for performing synthesis for three or four position rigid body guidance. It is important to note that in *S*PHINX the design sphere is of arbitrary radius and its location in space is undefined.

*S*PHINXPC (Ruth and McCarthy 1997) is a CAD program for personal computers which like *S*PHINX utilizes a design sphere with orientations displayed on the sphere's surface. With this software spherical mechanisms can be designed for four orientations. *S*PHINXPC also has the capability to design planar mechanisms for four position rigid body guidance.

In 1995 Osborn and Vance developed the first virtual reality(VR) based approach to spherical mechanism design, entitled *S*PHEREVR. This initial exploration of the use of VR for spherical mechanism design has led to the development of a 3rd generation of VR based spherical mechanism design software called

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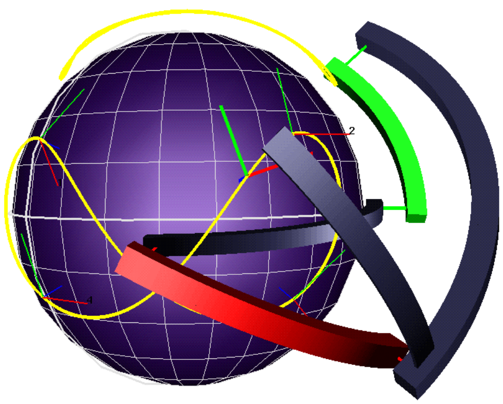

Figure 1. Sphinx DESIGN SPHERE

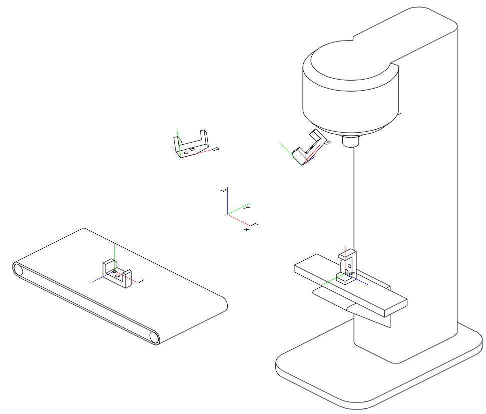

Figure 2. A DESIRED TASK

Isis, see Larochelle, Vance, and McCarthy 1998. The program utilizes the compute engine of Sphinx1.2 and provides virtual objects in the design environment so that the design process takes place in a virtual representation of the physical workspace. This new approach to mechanism design has demonstrated a need for new and efficient means for specifying the design task in the actual physical workspace of the mechanism.

To synthesize a spherical mechanism, the designer must first define the task to be accomplished. Here we are concerned with task specification for moving a workpiece through a sequence of prescribed orientations in space. This task is referred to as *rigid-body guidance* by Suh and Radcliffe 1978 and as *motion generation* by Erdman and Sandor 1997. An example of a rigid body guidance task is shown in Fig. 2. The desired positions of the workpiece are defined in space. A coordinate frame is attached to the workpiece and its location, in each of the desired positions, is recorded. To date, when designing spherical mechanisms the designer must determine an appropriate design sphere, i.e. its center and radius, from the desired spatial positions. Moreover, the sets of angles which define the orientations of the body with respect to that design sphere must also be determined. Currently, no methodologies exist to facilitate this process. It is only after determining the design sphere and the orientations that the designer can utilize CAD tools such as Sphinx and SphinxPC.

In this paper, one method of determining the optimal design sphere and orientations from a desired set of spatial positions is presented. First, the spatial positions are approximated with orientations in four-dimensional Euclidean space($\mathbf{E}^4$). Biquaternions are then used to represent these orientations. Next, the distance between the spatial positions and the orientations on a candidate design sphere are calculated using a bi-invariant metric on biquaternions. Finally, an optimization method is used to minimize the distances between the spherical orientations on the candidate design sphere and the spatial positions. The result is a procedure which numerically determines the optimal design sphere and orientations for a finite set of desired spatial positions.

ORIENTATIONS IN E^4 AND BIQUATERNIONS

In 1995 Larochelle and McCarthy presented an algorithm for approximating a set of n positions in planar Euclidean space ($\mathbf{E}^2$) with n spherical orientations in three-dimensional Euclidean space ($\mathbf{E}^3$). By utilizing a bi-invariant metric on the image space of spherical displacements they arrived at an approximate bi-invariant metric for planar positions in which the error induced by the spherical approximation is of the order $\frac{1}{R^2}$, where R is the radius of the approximating sphere. In this paper we extend their methodology to the general spatial case and utilize the results to provide a novel method of specifying motion generation tasks for spherical mechanisms.

It was shown in Larochelle and McCarthy 1995 that orientations in $\mathbf{E}^3$ may be used to approximate positions in a bounded region of a two-dimensional plane. We utilize the contributions of Etzel and McCarthy 1996 and extend that idea by using orientations in $\mathbf{E}^4$ to approximate positions in a bounded region of three-dimensional space. This can be done by using a small portion of a four-dimensional hypersphere, a *wedge*, to approximate a bounded region of space. Orientations on the surface of this wedge, which we represent with biquaternions, can be used to approximate the spatial positions. See Ge 1994 in which he examines the theory of biquaternions as representations of orientations on a hypersphere.

We proceed by briefly reviewing quaternions and biquaternions. Recall that an orientation in $\mathbf{E}^3$ can be represented by a quaternion $\mathbf{q} = [q_1\ q_2\ q_3\ q_4]^T$. The four components of the quaternion $\mathbf{q}$, sometimes referred to as Euler parameters are,

$$
\begin{aligned}
q_1 &= s_x \sin\frac{\theta}{2} = s_x s\frac{\theta}{2}\\
q_2 &= s_y \sin\frac{\theta}{2} = s_y s\frac{\theta}{2}\\
q_3 &= s_z \sin\frac{\theta}{2} = s_z s\frac{\theta}{2}\\
q_4 &= \cos\frac{\theta}{2} = c\frac{\theta}{2}
\end{aligned} \tag{1}
$$

where $\mathbf{s}$ and θ are the rotation axis and the angle of rotation associated with the orientation, respectively. Note that the components of $\mathbf{q}$ satisfy the following constraint equation,

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 - 1 = 0 \tag{2}$$

and lie on a unit hypersphere which we denote as *the image space of spherical displacements*.

Recall that the position of a body in $\mathbf{E}^3$ has six degrees of freedom (three to define orientation and three to define location) and can be represented by a 4x4 homogeneous transform(Paul 1981):

$$T = \begin{bmatrix} [R(\theta,\phi,\psi)] & \vdots\, \mathbf{d} \\ \cdots\cdots\cdots & \cdots \\ 0 \quad 0 \quad 0 & \vdots\, 1 \end{bmatrix} \tag{3}$$

$$[R(\theta,\phi,\psi)] = Rot_y(\theta) Rot_x(-\phi) Rot_z(\psi)$$

where $\mathbf{d}$ is a 3x1 translation vector. The angles θ, ϕ, and ψ are the longitude, latitude, and roll angles respectively (see Larochelle and McCarthy 1995). In 1996 Etzel and McCarthy showed that a 4x4 homogeneous transform in $\mathbf{E}^3$ can be approximated by a pure rotation in $\mathbf{E}^4$:

$$[D] = [J(\alpha,\beta,\gamma)][K(\theta,\phi,\psi)] \tag{4}$$

where,

$$J(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha & 0 & 0 & s\alpha \\ -s\beta s\alpha & c\beta & 0 & s\beta c\alpha \\ -s\gamma c\beta s\alpha & -s\gamma s\beta & c\gamma & s\gamma c\beta c\alpha \\ -c\gamma c\beta s\alpha & -s\beta c\gamma & -s\gamma & c\gamma c\beta c\alpha \end{bmatrix}$$

and,

$$K(\theta,\phi,\psi) = \begin{bmatrix} & \vdots\, 0 \\ [R(\theta,\phi,\psi)] & \vdots\, 0 \\ & \vdots\, 0 \\ \cdots\cdots\cdots & \cdots \\ 0 \quad 0 \quad 0 & \vdots\, 1 \end{bmatrix}.$$

The angles α, β and γ are defined as follows: $\tan(\alpha) = \frac{d_x}{\mathsf{R}}$, $\tan(\beta) = \frac{d_y}{\mathsf{R}}$, and $\tan(\gamma) = \frac{d_z}{\mathsf{R}}$ where d_x, d_y, and d_z are the components of $\mathbf{d}$ and R is the radius of the hypersphere.

The bounded spatial workspace must represent a only small portion of the hypersphere (referred to as a wedge), hence we determine the radius of the hypersphere as:

$$\mathsf{R} = \frac{4L}{\varepsilon^{\frac{1}{2}}} \tag{5}$$

where L is the largest component of the translation vectors from the set of spatial positions and ε is the maximum allowable error in the approximation of the spatial positions with the orientations in $\mathbf{E}^4$. Next, we review how to determine the biquaternion associated with the matrix $[D]$.

Recall that biquaternions have the following form:

$$\hat{\mathbf{G}} = \mathbf{G} + \omega\mathbf{H} \tag{6}$$

where $\mathbf{G}$ and $\mathbf{H}$ are quaternions and ω is defined such that $\omega^2 = 1$, see Ge 1994. The biquaternion can also be represented as an ordered pair of quaternions $\hat{\mathbf{G}} = (\mathbf{G},\mathbf{H})$. The quaternions $\mathbf{G}$ and $\mathbf{H}$ are determined by the following computations. The fourth components of $\mathbf{G}$ and $\mathbf{H}$ are $G_4 = \cos(\mu)$ and $H_4 = \cos(\nu)$ respectively, with μ and ν being the real part of the eigenvalues from matrix $[D]$. The other three components of $\mathbf{G}$ and $\mathbf{H}$ are computed as follows:

$$
\begin{aligned}
G_1 &= \tfrac{d_{23}-d_{32}+d_{14}-d_{41}}{4H_4}\\
G_2 &= -(\tfrac{d_{31}-d_{13}+d_{42}-d_{24}}{4H_4})\\
G_3 &= \tfrac{d_{21}-d_{12}+d_{34}-d_{43}}{4H_4}\\
H_1 &= \tfrac{d_{32}-d_{23}-d_{14}+d_{41}}{4G_4}\\
H_2 &= -(\tfrac{d_{31}-d_{13}-d_{42}+d_{24}}{4G_4})\\
H_3 &= \tfrac{d_{21}-d_{12}-d_{34}+d_{43}}{4G_4}
\end{aligned}
$$

where d_{ij} are the elements of $[D]$. From the above relations, it is evident that there are three special cases which need to be addressed, see Etzel 1996. First, if $G_4 = 0$ then the first three elements of $\mathbf{H}$ are:

$$H_1 = \frac{d_{11}+d_{44}}{2G_1}$$
$$H_2 = \frac{d_{22}+d_{44}}{2G_2}$$
$$H_3 = \frac{d_{33}+d_{44}}{2G_3}.$$

Second, if $H_4 = 0$ then the first three components of $\mathbf{G}$ are:

$$G_1 = \frac{d_{11}+d_{44}}{2H_1}$$
$$G_2 = \frac{d_{22}+d_{44}}{2H_2}$$
$$G_3 = \frac{d_{33}+d_{44}}{2H_3}.$$

Finally, if $G_4 = 0$ and $H_4 = 0$ then solve the following relations for H_i(i=1,2,3):

$$\frac{d_{21}-d_{43}}{H_2} = \frac{d_{11}+d_{44}}{H_1} = \frac{d_{31}+d_{42}}{H_3}$$

and obtain G_i as in the $H_4 = 0$ case above.

The Metric

There exist numerous useful metrics for defining the distance between two points in Euclidean space, however, defining similar metrics for determining the distance between two positions of a rigid body is still an area of ongoing research. In the case of two positions of a rigid body in $\mathbf{E}^3$ any metric used to measure the distance between the positions yields a result which depends upon the chosen reference frames, see Martinez and Duffy 1995. However, Ravani and Roth 1983 define the distance between two orientations in $\mathbf{E}^3$ as the magnitude of the difference between their associated quaternions, which is a bi-invariant metric. Recall that a bi-invariant metric is independent of choice of both the fixed and moving frames. Etzel and McCarthy 1996 extended this idea and presented a bi-invariant metric for orientations in $\mathbf{E}^4$. Here, we review their metric and present a methodology which employs the metric to determine the optimal design sphere associated with a finite set of spatial positions.

The bi-invariant metric on biquaternions is defined as:

$$d(\hat{\mathbf{Q}},\hat{\mathbf{R}}) = \sqrt{(\mathbf{Q}-\mathbf{R})^{\mathbf{T}}(\mathbf{Q}-\mathbf{R}) + (\mathbf{S}-\mathbf{T})^{\mathbf{T}}(\mathbf{S}-\mathbf{T})} \qquad (7)$$

where $\hat{\mathbf{Q}} = (\mathbf{Q},\mathbf{S})$ and $\hat{\mathbf{R}} = (\mathbf{R},\mathbf{T})$ are both biquaternions. For a proof that this metric is bi-invariant see Etzel and McCarthy 1996.

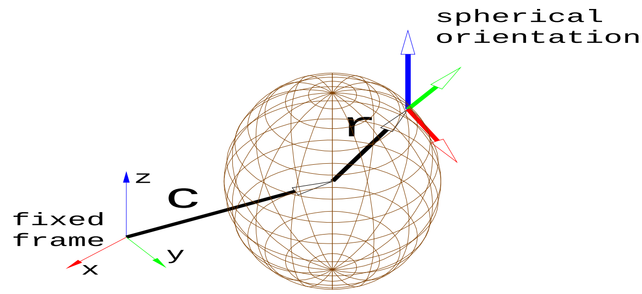

Figure 3. OPTIMAL DESIGN SPHERE

OPTIMIZING THE DESIGN SPHERE

In Fig. 3 a spherical orientation on a design sphere is shown. To obtain the orientation frame relative to the fixed frame three coordinate frame transformations are applied. First, the moving frame is translated along the 3x1 center vector $\mathbf{c}$. Next, the moving frame is rotated by the longitude, latitude, and roll angles as defined by Eq. 3. Third, the moving frame is translated along the 3x1 radial vector $\mathbf{r}$. The spherical orientation is now defined by the following 4x4 homogeneous transform:

$$T_{spherical}(\mathbf{r},\mathbf{c}) = \begin{bmatrix} [R] & \vdots & [R]\mathbf{r}+\mathbf{c} \\ \cdots & \cdots & \cdots \\ 0\ \ 0\ \ 0 & \vdots & 1 \end{bmatrix}$$

where $[R]$ is the 3x3 rotation matrix defined in Eq. 3. Let $T_{spatial}$ be the 4x4 homogeneous transform representation of a desired position of the workpiece in space. To determine the optimal design sphere the distance between $T_{spatial}$ and $T_{spherical}$ must be minimized for each of the n desired positions in $\mathbf{E}^3$. The next section presents a method to minimize this distance by utilizing the bi-invariant metric discussed above.

Optimization

Given a finite set of n desired positions in $\mathbf{E}^3$ the task is to determine the optimal design sphere and the n orientations on that sphere. By examining the homogeneous transform representation of $T_{spherical}$ it is clear that the optimization variables are $\mathbf{r}$ and $\mathbf{c}$ since $[R]$ may be extracted from $T_{spatial}$[1]. The optimization problem then becomes:

Minimize:

$$f(\mathbf{r},\mathbf{c})$$

[1]Note that by extracting $[R]$ in this manner we guarantee that the orientations of the n $T_{spherical}$ will be identical to that of their associated $T_{spatial}$.

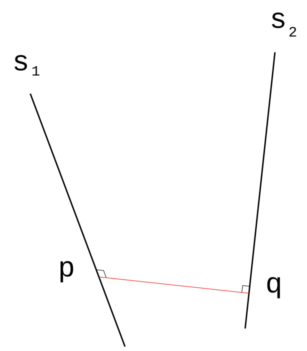

Figure 4. COMMON NORMAL OF TWO SCREW AXES

Subject to:

$$\| \mathbf{r} \| \leq 2L$$
$$\| \mathbf{c} \| \leq 2L$$

where:

$$f(\mathbf{r},\mathbf{c}) = \sum_{i=1}^{n} d(\hat{Q}_i,\hat{R}_i).$$

We utilize the *simplex method for function minimization* to find $\mathbf{r}$ and $\mathbf{c}$ that minimize $f(\mathbf{r},\mathbf{c})$, see Nelder and Mead 1965. This method was selected since it does not require analytical gradients and it is a direct multidimensional minimization algorithm.

Initialization

If the n spatial positions are in fact spherical orientations then the center of the design sphere is located at the intersection of the relative screw axes associated with the positions. However, with general spatial positions these relative screw axes will not intersect. Hence, we find the point nearest all of the relative screw axes and use it as the initial center of the optimal design sphere. In Fig. 4 the common normal associated with two relative screw axes is shown. The intersections of the common normal with the two screw axes are $\mathbf{p}$ and $\mathbf{q}$. Note that if the screw axes do not intersect then the point in space nearest the screw axes is the midpoint of the segment $\overline{\mathbf{pq}}$. The initial estimation of the center $\mathbf{c}$ is selected as the point nearest all of the relative screw axes associated with the spatial positions:

$$\mathbf{c}_{initial} = \frac{\sum_{i=1}^{l}\mathbf{p} + \sum_{i=1}^{l}\mathbf{q}}{2l} \tag{8}$$

where $l = \binom{m}{2}$ and $m = \binom{n}{2}$ is the number of relative screw axes[2].

The initialization of $\mathbf{r}$ is obtained by equating the translation vectors of $T_{spatial}$ and $T_{spherical}$. For any given spatial position the radial vector $\mathbf{r}$ of the design sphere is then,

$$\mathbf{r} = [R]^T(\mathbf{d}_{spatial} - \mathbf{c}). \tag{9}$$

Substituting $\mathbf{c}_{initial}$ into Eq. 9 we obtain:

$$\mathbf{r} = [R]^T(\mathbf{d}_{spatial} - \mathbf{c}_{initial}). \tag{10}$$

Using Eq. 10 we compute $\mathbf{r}$ for each spatial position. The initial estimation of the radial vector is then the average,

$$\mathbf{r}_{initial} = \frac{\sum_{i=1}^{n}\mathbf{r}}{n}. \tag{11}$$

Preserving One Position

It may be necessary for the designer to require that one of the desired $T_{spatial}$ be preserved. In this case the design sphere is constrained to exactly preserve this one spatial position(referred to as T_{exact}). The design sphere is then optimized to minimize the distance between the remaining $T_{spatial}$'s and their associated $T_{spherical}$'s. Let us label the elements of the 4x4 homogeneous transform representation of T_{exact} as,

$$T_{exact} = \begin{bmatrix} & [R_{exact}] & & \vdots\ \mathbf{d}_{exact} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots\ 1 \end{bmatrix}.$$

By equating the translation vectors of T_{exact} and $T_{spherical}$ we obtain:

$$\mathbf{d}_{exact} = [R_{exact}]\mathbf{r} + \mathbf{c}. \tag{12}$$

We note that Eq. 12 is a linear system of three equations in the six unknown components of $\mathbf{r}$ and $\mathbf{c}$. The simplex method for

[2]Note that $\binom{n}{r}$ denotes the binomial coefficient, often referred to as "*n choose r*".

function minimization is employed to optimize the location of the center of the design sphere **c** and Eq. 12 is used to determine **r** at each iteration,

$$\mathbf{r} = [R_{exact}]^T(\mathbf{d}_{exact} - \mathbf{c}). \quad (13)$$

SPHERICAL INDEX

Obviously, not all finite sets of general spatial positions can be approximated with spherical orientations. Some sets of spatial positions are more near spherical than others and yield better spherical approximations while other sets of spatial positions may be far from spherical and for these no acceptable spherical approximations exist.

The method presented here does not guarantee an acceptable set of spherical orientations may be found for every set of general spatial positions. Recall that the purpose of this method is to facilitate motion generation task specification for spherical mechanism design. The implication being that the set of spatial positions will be *near spherical* and the method we present here determines the exact spherical orientations which best approximate the near spherical positions. As a measure of how near spherical the original spatial positions are we utilize the following **spherical index** $\odot$:

$$\odot = \frac{\sum_{i=1}^{m} |d_{relative}|}{4Lm} \quad (14)$$

where $d_{relative}$ is the translation along the relative screw axes associated with two positions and m and L are as defined above. Sets of spatial positions with small $\odot$ yield acceptable spherical approximations while sets with large $\odot$ will not yield acceptable spherical approximations.

CASE STUDY

We now illustrate the task specification methodology by applying it to the motion generation task shown in Fig. 2. The longitude, latitude, and roll angles(in degrees) and translation vectors for the four desired spatial positions are found in Tbl. 1. The spherical index value for these positions is $\odot = 7.211E-8$ which indicates that these positions are very near spherical. Hence, we anticipate that there exist spherical orientations which are very near the original spatial positions and proceed with the numerical nonlinear optimization. The initial estimates of the center and radial vectors are $\mathbf{c}_{initial} = [0.2227\ 0.2218\ -0.1629]^T$ and $\mathbf{r}_{initial} = [-0.1084\ 0.2114\ 5.1736]^T$. The radius of the hypersphere is R = 2080, with $\varepsilon = 0.0001$ and $L = 5.2$. In Fig. 5 the optimal design sphere and orientations are shown. The spherical orientations are the position frames with thicker lines. The optimal center and radial vectors for this design sphere are $\mathbf{c} = [0.1019\ 0.0791\ 0.0244]^T$ and $\mathbf{r} = [-0.0771\ 0.0151\ 5.0821]^T$. The optimal orientations(1',2',3',4') and their distances from the original spatial positions are found in Tbl. 1.

Having now determined the orientations which best approximate the original spatial positions we can now use SPHINX to design a spherical four-bar mechanism to generate the desired motion. The resulting mechanism, as displayed by SPHINX, is shown in Fig. 6. In order to employ this design to generate the desired motion manufacture the coupler for a radius of $\| \mathbf{r} \|$, manufacture the remaining links at appropriate radii, mount the mechanism such that the center of its associated sphere is located at **c**, and attach the workpiece to the coupler.

SUMMARY

In this paper we have presented a novel method for motion generation task specification for spherical mechanisms. This was accomplished with a new methodology for determining the optimal design sphere and the orientations on this design sphere for a finite set of desired spatial positions. Moreover, we have included a modification to the algorithm such that one of the desired spatial positions is exactly preserved. The result is that mechanism designers can now specify spherical mechanism motion generation tasks without having to introduce into the design space an artificial design sphere. They are now free to work in unconstrained three-dimensional space.

Finally, we believe that the utility of this new task specification algorithm will be most evident when utilized in three-dimensional computer graphics design environments such as SPHINXPC and SPHINX. Moreover, we anticipate that it will be an asset to the new ISIS virtual reality spherical mechanism design environment currently being created in a collaborative effort lead by Dr. J.M. Vance at Iowa State and Dr. P.M. Larochelle at Florida Tech.

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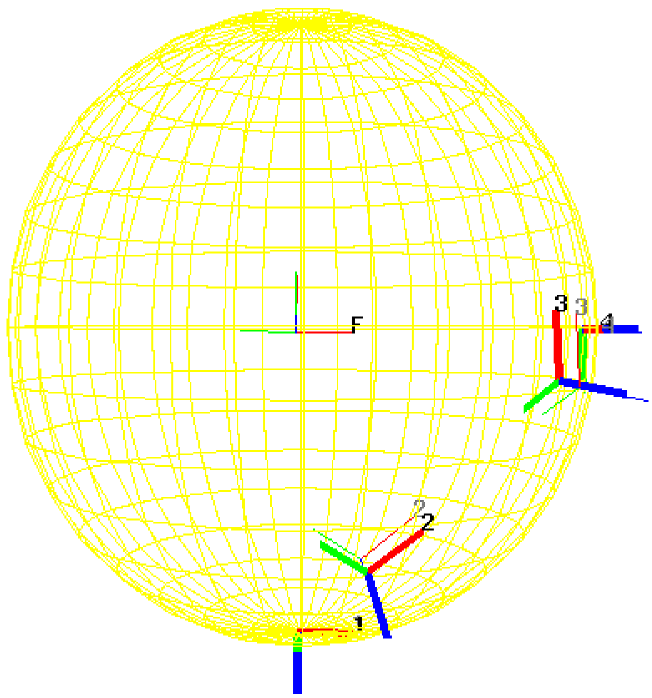

Figure 5. OPTIMAL DESIGN SPHERE AND ORIENTATIONS FOR THE DESIRED TASK

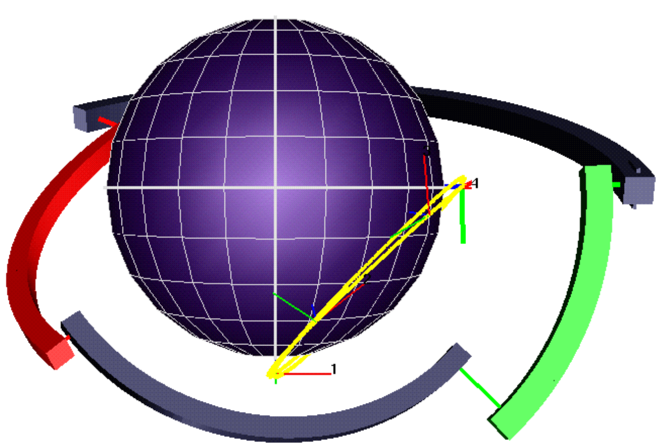

Figure 6. SPHERICAL MECHANISM FOR THE DESIRED TASK

Table 1. DESIRED SPATIAL POSITIONS AND THEIR ASSOCIATED OPTIMAL ORIENTATIONS

Pos.	Long.	Lat.	Roll	d_x	d_y	d_z	Distance
1	0.0	−90.0	0.0	0.0	−5.0	0.0	NA
1’	0.0	−90.0	0.0	0.02	−5.00	0.04	$1.555E-5$
2	14.12	−38.41	51.48	0.92	−2.98	3.65	NA
2’	14.12	−38.41	51.48	1.01	−3.12	3.87	$8.485E-6$
3	47.23	−7.46	108.55	4.00	−0.71	3.70	NA
3’	47.23	−7.46	108.55	3.80	−0.66	3.43	$1.555E-5$
4	90.0	0.0	180.0	5.2	0.0	0.0	NA
4’	90.0	0.0	180.0	5.18	0.06	−0.05	$1.697E-5$
TOTAL							$5.655E-5$

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