DETC98/MECH-5881

ALGEBRAIC MOTION APPROXIMATION WITH NURBS MOTIONS AND ITS APPLICATION TO SPHERICAL MECHANISM SYNTHESIS

Q. Jeffrey Ge
Department of Mechanical Engineering
State University of New York at Stony Brook
Stony Brook, NY 11794-2300, USA
ge@design.eng.sunysb.edu

Pierre M. Larochelle
Mechanical Engineering Program
Florida Institute of Technology
Melbourne, FL 32901-6988, USA
pierre@gretzky.me.fit.edu

ABSTRACT

In this work we bring together classical mechanism theory with recent works in the area of Computer Aided Geometric Design(CAGD) of rational motions as well as curve approximation techniques in CAGD to study the problem of mechanism motion approximation from a computational geometric viewpoint. We present a framework for approximating algebraic motions of spherical mechanisms with rational B-Spline spherical motions. Algebraic spherical motions and rational B-spline spherical motions are represented as algebraic curves and rational B-Spline curves in the space of quaternions (or the image space). Thus the problem of motion approximation is transformed into a curve approximation problem, where concepts and techniques in the field of Computer Aided Geometric Design and Computational Geometry may be applied. An example is included at the end to show how a NURBS motion can be used for synthesizing spherical four-bar linkages.

1 INTRODUCTION

Non-Uniform Rational B-Splines, commonly referred to as NURBS, have become the de facto industry standard for the representation, design, and data exchange of geometric information processed by computers. Recently, it has become apparent that NURBS geometry can be extended to kinematic domain for synthesizing NURBS motions of rigid bodies in Euclidean three-space (Ge and Ravani, 1994; Jüttler, 1994; Jüttler and Wagner, 1996; Ge and Kang, 1996; Ge et al., 1997). The purpose of the present paper is to present a framework for algebraic motion

approximation using spherical NURBS motion by combining NURBS geometry with kinematic geometry of spherical mechanisms. From the viewpoint of mechanism synthesis, the ideas presented in this paper are extensions of the work of Gupta and Roth (1975) on kinematic approximation of circles and straight lines, the series of work of Ravani and Roth (1983, 1984), Bodduluri and McCarthy (1992), Larochelle and McCarthy (1994) on algebraic motion synthesis using kinematic mapping, as well as the work of Liu and Angeles (1992a, 1992b) on planning global properties of a mechanism motion for optimization of function generating mechanisms.

The paper is organized as follows. Section 1 reviews how spherical displacements can also be represented projectively using homogeneous quaternions. Section 2 presents spherical rational Bézier and B-spline motions as Bézier and B-spline quaternion curves. Section 3 presents algebraic motions of spherical mechanisms. Section 4 discusses three motion approximation problems and presents an example to demonstrate the feasibility of our approach.

2 SPHERICAL DISPLACEMENTS

Quaternion algebra allows for an elegant treatment for spherical kinematics (Yang and Freudenstein, 1964; Ravani and Roth, 1984; Bottema and Roth, 1990; McCarthy, 1990). A quaternion is a hypercomplex number of the form $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} + q_4$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are quaternion units. The components q_i are

known as *Euler parameters* of rotation and are given by

$$q_1 = s_1 \sin\tfrac{\theta}{2}, \quad q_2 = s_2 \sin\tfrac{\theta}{2}, \quad q_3 = s_3 \sin\tfrac{\theta}{2}, \quad q_4 = \cos\tfrac{\theta}{2}, \quad (1)$$

where $\mathbf{s} = (s_1, s_2, s_3)$ is the unit vector along the axis of rotation and θ is the angle of rotation. Note that the Euler parameters satisfy the condition

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1, \tag{2}$$

and the corresponding quaternion is called a unit quaternion. In general, however, one can define a non-unit quaternion using the homogeneous Euler parameters, $\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4)$, where $Q_i = vq_i$ with $v > 0$.

Let the location of a point in Euclidean three-space E^3 before and after a spherical displacement be represented by homogeneous Cartesian vectors $\mathbf{p} = (p_1, p_2, p_3, p_4)$ and $\tilde{\mathbf{p}} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4)$, respectively. These homogeneous vectors define points in projective three-space P^3. One can associate these vectors with quaternions as well, which maybe referred to as *point quaternions*. The point coordinate transformation under a spherical displacement can be represented by the following quaternion product:

$$\tilde{\mathbf{p}} = \mathbf{Q}\mathbf{p}\mathbf{Q}^* \tag{3}$$

where $\mathbf{Q}^*$ denotes the conjugate of the rotation quaternion $\mathbf{Q}$ and $\tilde{\mathbf{p}}$, $\mathbf{p}$ are point quaternions[1]. The quaternion representation can be recast in 4×4 matrix form as

$$\tilde{\mathbf{p}} = [H(\mathbf{Q})]\mathbf{p} \tag{4}$$

where

$$[H(\mathbf{Q})] = [Q^+][Q^-]$$

with (see McCarthy, 1990; Ge, 1994)

$$[Q^+] = \begin{bmatrix} Q_4 & -Q_3 & Q_2 & Q_1 \\ Q_3 & Q_4 & -Q_1 & Q_2 \\ -Q_2 & Q_1 & Q_4 & Q_3 \\ -Q_1 & -Q_2 & -Q_3 & Q_4 \end{bmatrix}, \quad [Q^-] = \begin{bmatrix} Q_4 & -Q_3 & Q_2 & -Q_1 \\ Q_3 & Q_4 & -Q_1 & -Q_2 \\ -Q_2 & Q_1 & Q_4 & -Q_3 \\ Q_1 & Q_2 & Q_3 & Q_4 \end{bmatrix}. \tag{5}$$

---

[1]Note that we use boldface letters to denote both quaternions and vectors.

Two rotation quaternions, $\mathbf{Q} = v\mathbf{q}$ and $\mathbf{q}$, represent one and the same rotation since

$$[H(\mathbf{Q})] = [H(v\mathbf{q})] = v^2[H(\mathbf{q})].$$

The scalar v can thus be considered as a weighting factor for the rotation. Ravani and Roth (1984) considered the homogeneous Euler parameters $\mathbf{Q} = v\mathbf{q}$ as defining a point in a projective three-space, called the *image space* (denoted as Σ) of spherical kinematics. In this way, an algebraic curve in Σ corresponds to an algebraic motion and a polynomial curve in Σ corresponds to a rational motion.

It is important to point out that although points in E^3 as well as spherical displacements can be represented by points in projective three-space, the geometry of P^3 is considered to be flat while the geometry of Σ is considered to be spherical or elliptic. For example, the distance between two points $\mathbf{Q}_i$ and $\mathbf{Q}_{i+1}$ in Σ is defined as the angle between the two lines defined by $\mathbf{Q}_i$ and $\mathbf{Q}_{i+1}$, see Martinez and Duffy (1995) and Larochelle and McCarthy (1996).

3   SPHERICAL NURBS MOTIONS

Rational Bézier and B-spline curves, also known as NURBS[2] are standard topics in the field of Computer Aided Geometric Design (Farin, 1993). In this section, we consider the problem of defining a spherical motion such that its point trajectory is a spherical NURBS curve. The resulting motion is called a *spherical NURBS motion*. We first consider the case of rational Bézier spherical motions. We then discuss how the result can be extended to rational B-spline spherical motions.

Given a sequence of unit quaternions $\mathbf{q}_i$ as well as associated weights $v_i > 0$, one can construct homogeneous quaternions by $\mathbf{Q}_i = v_i\mathbf{q}_i$. Note that in order to take care of the problem that both $\mathbf{q}_i$ and $-\mathbf{q}_i$ correspond to the same spherical displacement, we choose the sign of $\mathbf{q}_i$ such that $\mathbf{q}_i \cdot \mathbf{q}_{i+1} \geq 0$ where the symbol "·" represents the usual vector dot product. A Bézier quaternion curve of degree n is given by

$$\mathbf{Q}^n(t) = \sum_{i=0}^{n} B_i^n(t)\mathbf{Q}_i. \tag{6}$$

The quaternions $\mathbf{Q}_i$ are here referred to as *Bézier control quaternions*. The Bézier polygon defined by the Bézier quaternions is an intrinsic control structure for the resulting motion. The control structure corresponds to a piecewise rotational motion[3].

---

[2]Non-Uniform Rational B-Splines.
[3]Each motion segment is a pure rotation about a fixed axis.

After substituting (6) into (3), we obtain the point trajectory of the corresponding motion as

$$\tilde{\mathbf{p}}^{2n}(t) = \sum_{i=0}^{n} \sum_{j=0}^{n} B_i^n(t) B_j^n(t) \mathbf{Q}_i \mathbf{p} \mathbf{Q}_j^*. \qquad (7)$$

The point trajectory can also be put in Bézier form as

$$\tilde{\mathbf{p}}^{2n}(t) = \sum_{k=0}^{2n} B_k^{2n}(t) \mathbf{a}_k \qquad (8)$$

where Bézier control points $\mathbf{a}_k$ are

$$\mathbf{a}_k = \frac{1}{C_k^{2n}} \sum_{i+j=k} C_i^n C_j^n \mathbf{Q}_i \mathbf{p} \mathbf{Q}_j^* \qquad (9)$$

and C_i^n are binomial coefficients. Thus a Bézier quaternion curve of degree n defines a rational Bézier spherical motion of degree $2n$, for its point trajectories are rational Bézier curves of degree $2n$.

Writing the Bézier control points $\mathbf{a}_k$ in matrix form, we obtain

$$\mathbf{a}_k = [H_k] \mathbf{p} \qquad (10)$$

where

$$[H_k] = \frac{1}{C_k^{2n}} \sum_{i+j=k} C_i^n C_j^n [Q_i^+][Q_j^-]. \qquad (11)$$

This leads to the following matrix representation of the rational Bézier motion as defined by the Bézier quaternion curve (6):

$$[H^{2n}(t)] = \sum_{k=0}^{2n} B_k^{2n}(t) [H_k]. \qquad (12)$$

Thus the matrices $[H_k]$ may be referred to as *Bézier control matrices*. These matrices are in general not orthogonal and thus represent affine displacements. They define an affine (or linear) control structure for the rational Bézier motion. The linear control structure for rational Bézier motions was first presented by Jüttler and Wagner (1996). The set of $(2n+1)$ Bézier control matrices $[H_k]$ are defined by $(n+1)$ Bézier control quaternions $\mathbf{Q}_i$.

A nth degree B-spline quaternion curve is given by

$$\mathbf{Q}^m(t) = \sum_{i=0}^{m} N_i^n(t) \mathbf{Q}_i, \qquad (13)$$

with $N_i^n(t)$ being the B-spline basis. It is not difficult to show that the point trajectories of the resulting motion are rational B-spline spherical curves of degree $2n$. Therefore a B-spline quaternion curve of degree n defines a rational Bézier spherical motion of degree $2n$. In producing an example for this paper, we have utilized cubic B-spline quaternion curves which have a piecewise Bézier form. The standard algorithm for converting the de-Boor points to the Bézier points is directly applicable to B-spline quaternion curves. The algorithm can also be inverted for cubic rational B-spline interpolation. These algorithms can be found in CAGD texts such as Farin (1993) and Piegl and Tiller (1995).

4 ALGEBRAIC SPHERICAL MOTIONS

Let us consider a spherical motion of the moving body such that two points, $\mathbf{p}_1$ and $\mathbf{p}_2$, of the body trace out two separate algebraic curves on the surface of a unit sphere in E^3. These algebraic curves are given by the following algebraic equations:

$$f_i(\tilde{\mathbf{p}}_i; \mathbf{r}_i) = 0, \quad i = 1, 2, \qquad (14)$$

where $\mathbf{r}_i$ denote the coefficients or shape parameters of the algebraic curve $f_i = 0$. The equations $f_i = 0$ are homogeneous in $\mathbf{p}_i$, i.e. we have $f_i(z\tilde{\mathbf{p}}_i) = z^{k_i} f_i(\tilde{\mathbf{p}}_i)$ where z is a nonzero scalar and k_i is an integer. Substituting (4) into (14), we obtain the following two homogeneous equations in $\mathbf{Q}$:

$$F_i(\mathbf{Q}; \mathbf{p}_i, \mathbf{r}_i) = 0, \quad i = 1, 2 \qquad (15)$$

Each of the two equations defines an algebraic surface in the image space Σ and represents the set of all possible spherical displacements that satisfy the algebraic constraint (14). The intersection of the two surfaces defines an algebraic quaternion curve in Σ. This algebraic quaternion curve corresponds to an algebraic spherical motion such that any point of the moving body traces out an algebraic path.

For instance, the quaternion curve representing a spherical four-bar motion can be defined this way. Let $\mathbf{p}_i = (p_{i,1}, p_{i,2}, p_{i,3}, p_{i,4})$ be the homogeneous vector representing the locations of the moving pivots on the moving unit sphere, and let $\mathbf{m}_i = (m_{i,1}, m_{i,2}, m_{i,3}, m_{i,4})$ be homogeneous vector representing the locations of the fixed pivots on the fixed unit sphere. Let ρ_i denote the angular lengths of the driving and driven links. The

3

moving pivots are required to stay on circular paths, which are given by

$$f_i = p_{i,1}m_{i,1} + p_{i,2}m_{1,2} + p_{i,3}m_{i,3} - p_{i,4}m_{i,4}\cos\rho_i = 0, \quad i = 1, 2. \tag{16}$$

The corresponding quaternion surfaces in Σ are given by the following homogeneous quadric equations

$$\mathbf{Q}^T[F_i(\mathbf{p}_i, \mathbf{m}_i)]\mathbf{Q} = 0, \quad i = 1, 2, \tag{17}$$

where the coefficient matrices $[F_i(\mathbf{p}_i, \mathbf{m}_i)]$ are given by

$$[F_i] = \begin{bmatrix} p_{i,1}m_{i,1} - p_{i,2}m_{i,2} - p_{i,3}m_{i,3} - p_{i,4}m_{i,4}\cos\rho_i \\ p_{i,2}m_{i,1} + p_{i,1}m_{i,2} \\ p_{i,3}m_{i,1} + p_{i,1}m_{i,3} \\ p_{i,2}m_{i,3} - p_{i,3}m_{i,2} \\ p_{i,2}m_{i,1} + p_{i,1}m_{i,2} \\ -p_{i,1}m_{i,1} + p_{i,2}m_{i,2} - p_{i,3}m_{i,3} - p_{i,4}m_{i,4}\cos\rho_i \\ p_{i,3}m_{i,2} + p_{i,2}m_{i,3} \\ p_{i,3}m_{i,1} - p_{i,1}m_{i,3} \\ p_{i,3}m_{i,1} + p_{i,1}m_{i,3} \\ p_{i,3}m_{i,2} + p_{i,2}m_{i,3} \\ -p_{i,1}m_{i,1} - p_{i,2}m_{i,2} + p_{i,3}m_{i,3} - p_{i,4}m_{i,4}\cos\rho_i \\ p_{i,1}m_{i,2} - p_{i,2}m_{i,1} \\ p_{i,2}m_{i,3} - p_{i,3}m_{i,2} \\ p_{i,3}m_{i,1} - p_{i,1}m_{i,3} \\ p_{i,1}m_{i,2} - p_{i,2}m_{i,1} \\ p_{i,1}m_{i,1} + p_{i,2}m_{i,2} + p_{i,3}m_{i,3} - p_{i,4}m_{i,4}\cos\rho_i \end{bmatrix} . \tag{18}$$

The intersection of these two surfaces in Σ is the image curve of a spherical four-bar motion. The curve is a quartic algebraic curve of the first kind and the topological structure of the curve is directly related to the linkage type such as Grashof, non-Grashof, foldable linkages (Ge and McCarthy, 1991; Chase and Mirth, 1993).

5 ALGEBRAIC MOTION APPROXIMATION

In the image space Σ, the kinematic problem of algebraic motion approximation becomes a geometric problem of curve fitting in Σ. In this section, we discuss the following three problems related to algebraic motion approximation:

1. Approximation of a given spherical four-bar motion with a NURBS motion;
2. Approximation of a NURBS motion with a spherical four-bar motion;
3. Constrained NURBS motion approximation. The NURBS motion is required to fit a set of given spherical displacements while maintaining the kinematic structure of a spherical four-bar motion.

All three problems can be solved as curve-fitting problems in the Image Space Σ.

Essential to the motion approximation process is the estimation of the approximation error between the NURBS motion and the four-bar motion. There are two ways to estimate the error in the image space Σ. One is to calculate a sequence of points on the B-spline quaternion curve and then estimate the normal distance from each point to the image curve of a spherical four-bar motion. Calculation of a sequence of points on a B-spline curve is a routine task in computer graphics for rendering the curve. It can be done efficiently and reliably. The problem of calculating the normal distance from a point to the image curve is more challenging but has been effectively solved by Ravani and Roth (1983, 1984). The resulting algebraic curve-fitting technique has been refined and extended by Bodduluri and McCarthy (1992), Ge and Ravani (1993), and Larochelle and McCarthy (1994).

Another way to estimate the approximation error is to generate a sequence of points on the image curve of the four-bar motion and then estimate the normal distance from each point to the B-spline quaternion curve. Calculation of a sequence of points on an algebraic curve is in general more difficult than that for a parametric curve due to the algebraic form as well as the topological structure (such as possible multiple branches and self intersections) of the algebraic curve (Arnon, 1983). In the case of the image curve of a spherical mechanism's motion, the problem of calculating a sequence of points on the curve is equivalent to the position analysis of the mechanism motion and can be solved using the loop-closure equations of the mechanism (see McCarthy, 1990). Once a sequence of image points has been generated, one can take advantage of the convex-hull and subdivision property of a B-spline curve to develop reliable and efficient methods for estimating the error. Since the metric geometry of Σ is spherical "flat" algorithms in CAGD for distance calculation have to be modified to take into account the geometry of Σ.

The solution to the first problem would provide an *approximate piecewise rational parameterization* for algebraic motions of a spherical four-bar. This problem can be solved with a number of NURBS curve fitting techniques in CAGD including interpolation and approximation (see Chapter 9 of Piegl and Tiller, 1995).

In the second problem, we use a NURBS motion to plan a desired motion to include global properties such as a Grashof linkage or to eliminate order or branch defect problems. Liu and Angeles (1992a, 1992b) were probably the first who used spline curves to plan input-output curves for optimization of function generating mechanisms. Since the topological structure of an algebraic quaternion curve of a spherical four-bar motion has been classified and has been shown to be directly related the linkage type, one can plan a NURBS motion to capture desired topological structure of a desired four-bar motion. After a B-spline quaternion curve has been planned, one can obtain an approximating algebraic quaternion curve using the curve-fitting tech-

nique developed by Ravani and Roth and refined by McCarthy, Bodduluri, and Larochelle.

The quality of the above approximation is expected to be dependent on the shape of the given NURBS spherical motion. This is because while NURBS curves are free-form curves that can be used to model any curve shape, the image curve of a spherical four-bar is constrained by the kinematic structure of a four-bar closed-chain. This gives rise to the third problem in algebraic motion approximation, i.e. how to fit a small number of data points with a NURBS motion while maintaining the kinematic constraints of a four-bar motion. This problem may be solved by first finding a NURBS motion that fits the given data with more control points than necessary and then determining the extra control points such that the final NURBS curve fits the kinematic constraints of a four-bar motion.

Figure 1. A spherical four-bar linkage that approximates a set of 10 spherical displacements.

We now present an example that demonstrates the feasibility of our approach. First, we synthesize a spherical four-bar that approximates ten coupler positions. The resulting linkage is a non-Grashof double rocker with an average position error of 0.0039 and is shown in Figure 1. We then used a NURBS motion to interpolate the 10 coupler positions and generated 37 coupler positions on the NURBS motion and designed a mechanism for these 37 positions. The resulting mechanism, shown in Figure 2, is a non-Grashof double rocker with an average position error of 0.0062. Link lengths for both solutions are listed in Table 1. Note that by examining the coupler curve shown in Figure 1 we see that the 10 position solution mechanism suffers from order defect (the positions are reached in the order: 1-6, 10, 9, 8, 7). However, the 37 position solution mechanism does not suffer from order defect. The additional 27 positions may be viewed as nine sets of 3 positions, each of these sets being used to describe the desired coupler motion between two of the orig-
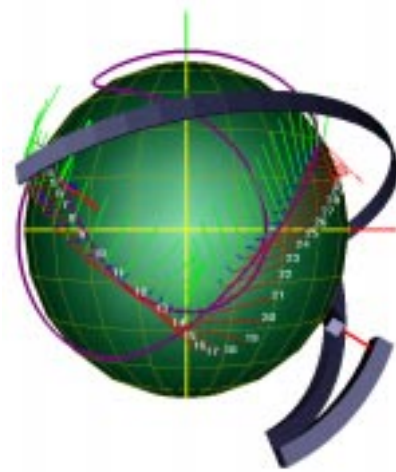

Figure 2. A spherical four-bar linkage that approximates a set of 37 spherical displacements.

inal positions. For example, in the 37 position case, positions 2 − 4 describe the desired motion from position 1 to position 2 of the 10 original positions. The result is the elimination of the order defect in this example.

CONCLUSIONS

In this paper we presented a framework for combining recent developments in the fields of Computer Aided Geometric Design with classical kinematic geometry of spherical motions and mechanisms to study the problem of spherical motion approximation from a computational geometric viewpoint. A quaternion-based representation of spherical displacements is used to transform the kinematic problem of motion synthesis into a geometric problem of curve design. In this way, algebraic motions of spherical mechanisms are represented by algebraic quaternion curves and NURBS spherical motions are represented by B-spline quaternion curves. The problem of algebraic motion approximation is studied as that of algebraic curve approximation in the space of quaternions. The initial ideas presented here forms a basis for future research in developing computational-geometric methods for mechanism design and analysis.

ACKNOWLEGEMENT

REFERENCES

Mechanism Link Parameters		
	Length (deg)	
Link	10 Positions	37 Positions
DRIVING	141.57	39.55
COUPLER	125.15	100.84
DRIVEN	44.12	34.91
FIXED	67.76	96.11

Table 1. Spherical $4R$ Synthesis: Design Results

Arnon, D.S., 1983, "Topologically reliable display of algebraic curves", *ACM Computer Graphics*, 17(3):219-227.

Bodduluri, R.M.C., and McCarthy, J.M., 1992, "Finite position synthesis using image curve of a spherical four-bar motion", *ASME J. of Mechanical Design*, 114(1):55-60.

Bottema, O., and Roth, B., 1990, *Theoretical Kinematics*, Dover Publ., New York.

Chase, T. R., and Mirth, J. A., 1993, Circuits and Branches of Single-Degree-of-Freedom Planar Linkages. *ASME J. of Mechanical Design*, Vol. 115, pp. 223-30.

Farin, G., 1993. *Curves and Surfaces for Computer Aided Geometric Design*. 3rd ed., Academic Press.

Ge, Q. J., and McCarthy, J. M., 1991, "The algebraic classification of the image curves of spherical four-bar motion", ASME Journal of Mechanical Design, 113(3):227-231.

Ge, Q.J., and Ravani, B., 1993, "Motion interpolation and mechanism synthesis". *Proc. 1993 ASME Design Automation Conference*.

Ge, Q.J., and Ravani, B., 1994, "Computer aided geometric design of motion interpolants". *ASME Journal of Mechanical Design*, 116(3):756-762.

Ge, Q.J., 1994, "On the matrix algebra realization of the theory of biquaternions," In *Mechanism Synthesis and Analysis*, volume ASME Publication DE-Vol. 70, pages 425–432.

Ge, Q. J., and Kang, D., 1996, "Kinematics-Based Geometric Design of Sculptured Surfaces", *Proc. 1996 ASME Design Manufacturing Conference*, Irvine, CA.

Gupta, K.C., and Roth, B., 1975, "A general approximation theory for mechanism synthesis", *ASME J. of Applied Mechanics*, June, 1995, pp. 451-457.

Jüttler, B., and Wagner, M.G., 1996, *Computer Aided Design With Spatial Rational B-Spline Motions*, *ASME J. of Mechanical Design*, 119(2):193-201.

Larochelle, P., and McCarthy, J.M., 1994, "Design of spatial 4C mechanisms for rigid body guidance", *Proc. 1994 ASME Mechanisms Conference*, p 135-142.

Larochelle, P. M. and McCarthy, J. M., 1996, Designing planar mechanisms using a bi-invariant metric in the image space of SO(3), *ASME J. of Mechanical Design*, Vol 117, pp. 646-651.

Liu, Z., and Angeles, J., 1992a, "Optimization of function-generating mechanisms using input-output curve planning: Part I – Curve planning and data preparation", *Proc. 1992 ASME Mechanisms Conference*, DE-Vol. 46, pp. 545-550.

Liu, Z., and Angeles, J., 1992b, "Optimization of function-generating mechanisms using input-output curve planning: Part II – Formulation and optimization strategies", *Proc. 1992 ASME Mechanisms Conference*, DE-Vol. 46, pp. 557-563.

Martinez, J.M.R. and Duffy, J., 1995, On the metrics of rigid body displacements for infinite and finite bodies, *ASME Journal of Mechanical Design*, Vol 117, pp. 41-47.

McCarthy, J. M., 1990, *Introduction to Theoretical Kinematics*. MIT Press, Cambridge, Massachusetts, 130pp.

Ravani, B., and Roth, B., 1983, "Motion synthesis using kinematic mappings", *ASME J. of Mech., Trans., and Auto. in Des.*, 105(3):460-467.

Ravani, B., and Roth, B., 1984, "Mappings of spatial kinematics", *ASME J. of Mech., Trans., and Auto. in Des.*, 106(3):341-347.

Shoemake, K., 1985, "Animating rotation with quaternion curves," *ACM Computer Graphics*, 19(3):245-254.

Yang, A. T., and Freudenstein, F., 1964, "Application of dual-number quaternion algebra to the analysis of spatial mechanisms," *J. of Applied Mechanics*, 31:300–308, 1964.