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SYNTHESIS OF PLANAR RR DYADS BY CONSTRAINT MANIFOLD PROJECTION

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ABSTRACT

In this paper we present the constraint manifold of the planar RR dyad. The constraint manifold is an analytical representation of the workspace of the dyad. We then derive a technique, utilizing the constraint manifold, for performing the dimensional synthesis of planar RR dyads for approximate rigid body guidance through n positions. Finally, we present the implementation of the design methodology in the software *VISSYN* and discuss its use in a design case study.

INTRODUCTION

The constraint manifold of a dyad represents the geometric constraint imposed on the motion of the moving body. This geometric constraint on the moving body is a result of the kinematic structure of the dyad; e.g. its length and the location of its fixed and moving axes. The constraint manifold is an analytical representation of the workspace of the dyad which is parameterized by the dyad's dimensional synthesis variables. Here we derive the constraint manifold of planar RR dyads in the image space of planar displacements and utilize this constraint manifold to perform dyadic dimensional synthesis for approximate rigid body guidance.

The derivation of the constraint manifold in the image space involves writing the kinematic constraint equations of the dyad using the components of a planar quaternion. We view these equations as constraint manifolds in the image space of planar displacements, see Bottema and Roth (1979) and McCarthy (1990). The result is an analytical representation of the workspace of the dyad which is parameterized by its dimensional synthesis variables. The synthesis goal is to vary the design variables such that all of the prescribed positions are either: (1) in the workspace, or, (2) the workspace comes as close as possible to all of the desired positions. Recall that in general five is the largest number of positions for which an exact solution is possible, see Suh and Radcliffe (1978). Previous works discussing constraint manifold fitting for an arbitrary number of positions include Ravani and Roth (1983), Bodduluri and McCarthy (1992), Bodduluri (1990), and Larochelle (1994). These works sought to use numerical non-linear optimization techniques to minimize some distance measure from the constraint manifold to the desired positions. These efforts are difficult to implement due to the highly non-linear nature of the problem. For example, when solving for a spherical fourbar mechanism for 10 positions Bodduluri and McCarthy (1992) utilized 120 starting cases of which 38 converged to the solution.

The synthesis procedure presented here involves the projection of the constraint manifold onto $\Re^3$ and using three-dimensional computer graphics to visualize the projected manifold. The designer then varies the dimensional synthesis variables until the constraint manifold satisfies the synthesis goals. In what follows, we apply the synthesis procedure to planar RR dyads and we illustrate its application in a case study.

IMAGE SPACE OF PLANAR DISPLACEMENTS

First, we review the use of planar quaternions for describing planar rigid-body displacements. A general planar displacement, occurring in the $X-Y$ plane, may be de-

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scribed bya 3×3 orthonormal rotation matrix $[A]$ and a translation vector $\mathbf{d} = [d_x\ d_y\ 1]^T$. Associated with the matrix of rotation $[A]$ is an axis of rotation $\mathbf{s} = [0\ 0\ 1]^T$ and a rotation angle θ.

Using the translation vector $\mathbf{d}$ and the rotation angle θ, we can represent a planar displacement by the four dimensional vector $\mathbf{q}$ which is written as, see McCarthy (1990),

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \frac{d_x}{2}\cos\frac{\theta}{2} + \frac{d_y}{2}\sin\frac{\theta}{2} \\ -\frac{d_x}{2}\sin\frac{\theta}{2} + \frac{d_y}{2}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix} \tag{1}$$

We refer to $\mathbf{q}$ as a planar quaternion. The components of $\mathbf{q}$ satisfy,

$$G(\mathbf{q}) : q_3^2 + q_4^2 - 1 = 0 \tag{2}$$

Note that $\mathbf{q}$, given by Eq. 1, is a four dimensional vector which satisfies the constraint equation, Eq. 2, therefore, the components of $\mathbf{q}$ form a three dimensional algebraic manifold. We denote this manifold as *the image space of planar displacements.*

Planar Quaternion Product

Given two planar quaternions, $\mathbf{g}$ and $\mathbf{h}$, their product yields a planar quaternion which represents the planar displacement obtained by the successive application of the two given displacements. We may write the product of two planar quaternions in the following matrix form, see McCarthy (1990),

$$\mathbf{gh} = G^{+}\mathbf{h} = H^{-}\mathbf{g} \tag{3}$$

where,

$$G^{+} = \begin{bmatrix} g_4 & -g_3 & g_2 & g_1 \\ g_3 & g_4 & -g_1 & g_2 \\ 0 & 0 & g_4 & g_3 \\ 0 & 0 & -g_3 & g_4 \end{bmatrix}$$

and,

$$H^{-} = \begin{bmatrix} h_4 & h_3 & -h_2 & h_1 \\ -h_3 & h_4 & h_1 & h_2 \\ 0 & 0 & h_4 & h_3 \\ 0 & 0 & -h_3 & h_4 \end{bmatrix}$$

CONSTRAINT MANIFOLD

In this section we derive the constraint manifold of the planar RR dyad. This dyad may be combined serially to form a complex open chain or, when connected back to the fixed link, may be joined so as to form a closed chain; e.g. a planar four-bar mechanism.

The constraint manifold is derived by using the geometric conditions that the joints of the dyad impose on the moving body. The structure equations for the geometric constraints are based upon the work of Ge (1990), McCarthy (1990), Suh and Radcliffe (1978), and Bodduluri (1990). Using the image space representation of planar displacements and the geometric constraint equations of the dyad we arrive at constraint equations in the image space that are parameterized by the dimensional synthesis variables of the dyad.

A planar RR dyad of length a is shown in Fig. 1. Let the axis of the fixed joint be specified by the vector $\mathbf{u}$ measured in the fixed reference frame F and let the origin of the moving frame be specified by $\mathbf{v}$ measured in the link frame A. The dimensional synthesis variables of the dyad are $\mathbf{u}$, $\mathbf{v}$, a, and ψ. Moreover, let us define the vector $\boldsymbol{\lambda}$ as representing the moving axis as measured in the moving frame M. The vectors $\mathbf{v}$ and $\boldsymbol{\lambda}$ are related by,

$$\boldsymbol{\lambda} = -[A(\psi)]^T\mathbf{v} \tag{4}$$

where ψ is the angle prescribing the orientation of M with respect to A.

We obtain the structure equation in the image space of planar displacements by using planar quaternions to represent the displacements from F to M,

$$\mathbf{d} = \mathbf{x}(u_x)\mathbf{y}(u_y)\mathbf{z}(\theta)\mathbf{x}(a)\mathbf{z}(\phi)\mathbf{x}(v_x)\mathbf{y}(v_y)\mathbf{z}(\psi) \tag{5}$$

where $\mathbf{x}(\cdot)$, $\mathbf{y}(\cdot)$, and $\mathbf{z}(\cdot)$ are planar quaternion representations of displacements either along or about the X, Y, or Z axes respectively.

We now rewrite $\mathbf{d}$ as,

$$\mathbf{d} = \mathbf{g}\mathbf{d}'\mathbf{h} \tag{6}$$

where: $\mathbf{g}$ is the displacement from F to O, where O is the frame with origin at the fixed pivot as shown in Fig. 1, $\mathbf{g} = \mathbf{x}(u_x)\mathbf{y}(u_y)$, $\mathbf{h}$ is the displacement frame A to M, $\mathbf{h} = \mathbf{x}(v_x)\mathbf{y}(v_y)\mathbf{z}(\psi)$ and $\mathbf{d}'$ is the displacement along the dyad, $\mathbf{d}' = \mathbf{z}(\theta)\mathbf{x}(a)\mathbf{z}(\phi)$. Performing the quaternion multi-

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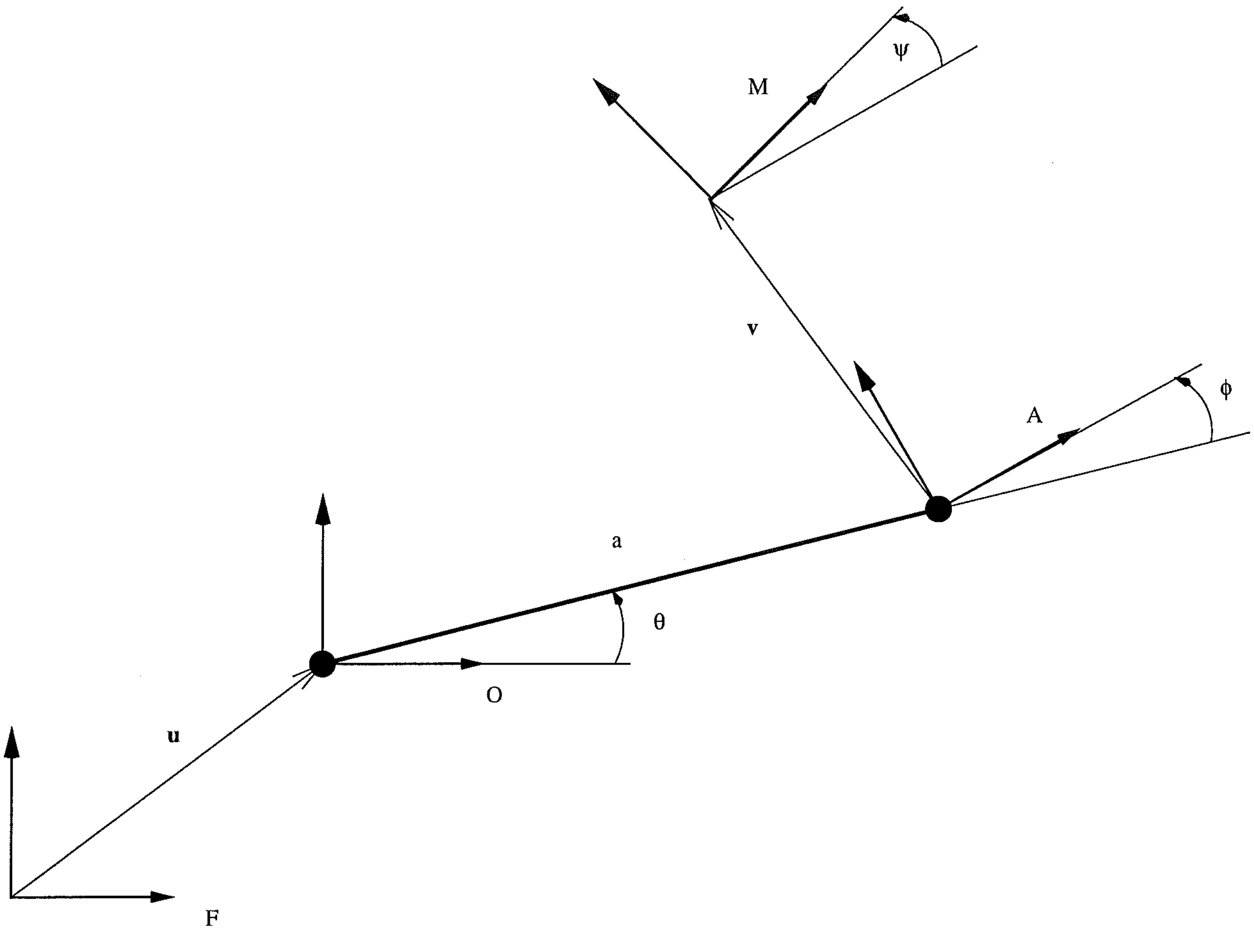

Figure 1. A Planar RR Dyad

plications yields,

$$\mathbf{d}' = \begin{bmatrix} \frac{a}{2}\cos\frac{\theta-\phi}{2} \\ \frac{a}{2}\sin\frac{\theta-\phi}{2} \\ \sin\frac{\theta+\phi}{2} \\ \cos\frac{\theta+\phi}{2} \end{bmatrix} \qquad (7)$$

$$\mathbf{g} = \begin{bmatrix} \frac{u_x}{2} \\ \frac{u_y}{2} \\ 0 \\ 1 \end{bmatrix} \qquad (8)$$

and,

$$\mathbf{h} = \begin{bmatrix} \frac{v_y}{2}\sin\frac{\psi}{2} + \frac{v_x}{2}\cos\frac{\psi}{2} \\ \frac{v_y}{2}\cos\frac{\psi}{2} - \frac{v_x}{2}\sin\frac{\psi}{2} \\ \sin\frac{\psi}{2} \\ \cos\frac{\psi}{2} \end{bmatrix} \qquad (9)$$

Finally, using Eq. 3 we express Eq. 6 as,

$$\mathbf{d} = \mathbf{c}\mathbf{d}' = G^{+}H^{-}\mathbf{d}' \qquad (10)$$

In Eq. 10 we have a surface in the image space of planar displacements which is parameterized by the dimensional synthesis variables of the dyad. This surface is the constraint manifold of the planar RR dyad.

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PROJECTION ONTO $\Re^3$

In the previous section we obtained the constraint manifold of the planar RR dyad in the image space of planar displacements. The image space is a three dimensional algebraic manifold on $\Re^4$. We now project the constraint manifold in the image space onto $\Re^3$.

We identify the first three planar quaternion coordinates of points in the constraint manifold with the coordinates of a point **cm** in $\Re^3$ as follows,

$$\mathbf{cm} = \begin{bmatrix} cm_1 \\ cm_2 \\ cm_3 \end{bmatrix} \iff \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad (11)$$

where d_1, d_2, and d_3 are the first three components of the planar quaternion **d** given by Eq. 10. The locus of all points **cm** is the projection of the workspace of the dyad onto $\Re^3$. Moreover, we note that the mapping given by Eq. 11 is one-to-one. Hence, associated with each position of the moving body M is a point **cm** in $\Re^3$ which is the projection of the point **d** in the image space given by Eq. 10.

Note that in the special case in which $\mathbf{c} = \mathbf{g} = \mathbf{h} = [0\ 0\ 0\ 1]^T$, that is to say frame O is coincident with frame F and frame M is coincident with frame A (i.e. $u_x = u_y = v_x = v_y = \psi = 0$), the locus of all points **cm** is a right circular cylinder of length 2, radius $\frac{a}{2}$, and with major axis along the Z axis, see Fig. 2. For the cases in which $\mathbf{g} \neq 0$ or $\mathbf{h} \neq 0$ we see from Eq. 10 that the right circular cylinder undergoes a coordinate transformation **c** which distorts the cylinder but that the lines of constant θ and ϕ remain intact.

SYNTHESIS

We proceed by utilizing the projection of the constraint manifold onto $\Re^3$ given by Eq. 11 to perform approximate motion synthesis. Utilizing the coordinate assignments found in Fig. 1 the design vector **r** for a planar RR dyad is,

$$\mathbf{r} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ a \\ \psi \end{bmatrix} \quad (12)$$

The goal of the synthesis procedure is to determine **r** such that the moving body moves as close as necessary to the n prescribed positions. First, we project the constraint manifold of an initial dyad onto $\Re^3$. Next, we determine the image point associated with each of the desired positions and project these points onto $\Re^3$. Hence, each desired position of the moving body is represented by a point in $\Re^3$. We then interactively examine $\Re^3$ and vary the components of **r** until the projected constraint manifold passes through, or comes as close as necessary, to all of the n desired points(i.e. the desired positions). Since this is a highly visual synthesis procedure it is best understood by the series of images found in the case study below.

Finally, several planar RR dyads may be combined serially to form complex open chain robots or, when connected back to the fixed link, they may be joined so as to form a closed chain; i.e. a planar four bar mechanism. For these closed chains the constraint manifold of the mechanism is the intersection of the constraint manifolds of its open subchains. For example, the constraint manifold of a planar four bar mechanism is the the intersection of the constraint manifolds of its driving and driven dyads. Hence, we may synthesize closed chain mechanisms by varying their synthesis variables until the intersection of their projected subchain constraint manifolds passes through, or comes as close as necessary, to all of the n desired projected positions.

VISSYN

The synthesis procedure described above has been implemented in *C* using the *GL* graphics library on a Silicon Graphics Indigo^2 UNIX based workstation in the program *VISSYN* [1] (VISual SYNthesis).

A design session using *VISSYN* proceeds as follows. Upon execution, the n desired positions and **r** for the initial dyad are read from a data file. The projected constraint manifold is then displayed. Next, centered at each of the projected desired positions a solid sphere is drawn, see Fig. 3. The user then proceeds to interactively vary the dimensional synthesis variables (i.e. u_x, u_y, v_x, v_y, a, and ψ) while visually inspecting the projected constraint manifold. This process continues until the constraint manifold passes satisfactorily near the n projected desired positions.

In order to aid the designer in visualizing the physical design space of the dyad a two-dimensional display of the current dyad is available, see Fig. 4. The displays of the constraint manifold and of the design space of the dyad are updated continuously as the design variables are manipulated. Moreover, the distance in the image space from the constraint manifold to each desired position is digitally displayed in the two-dimensional display. This distance is computed using the metric presented by Ravani and Roth (1983), Bodduluri (1990), and Larochelle (1995).

[1] *VISSYN* will be made available by the author upon request.

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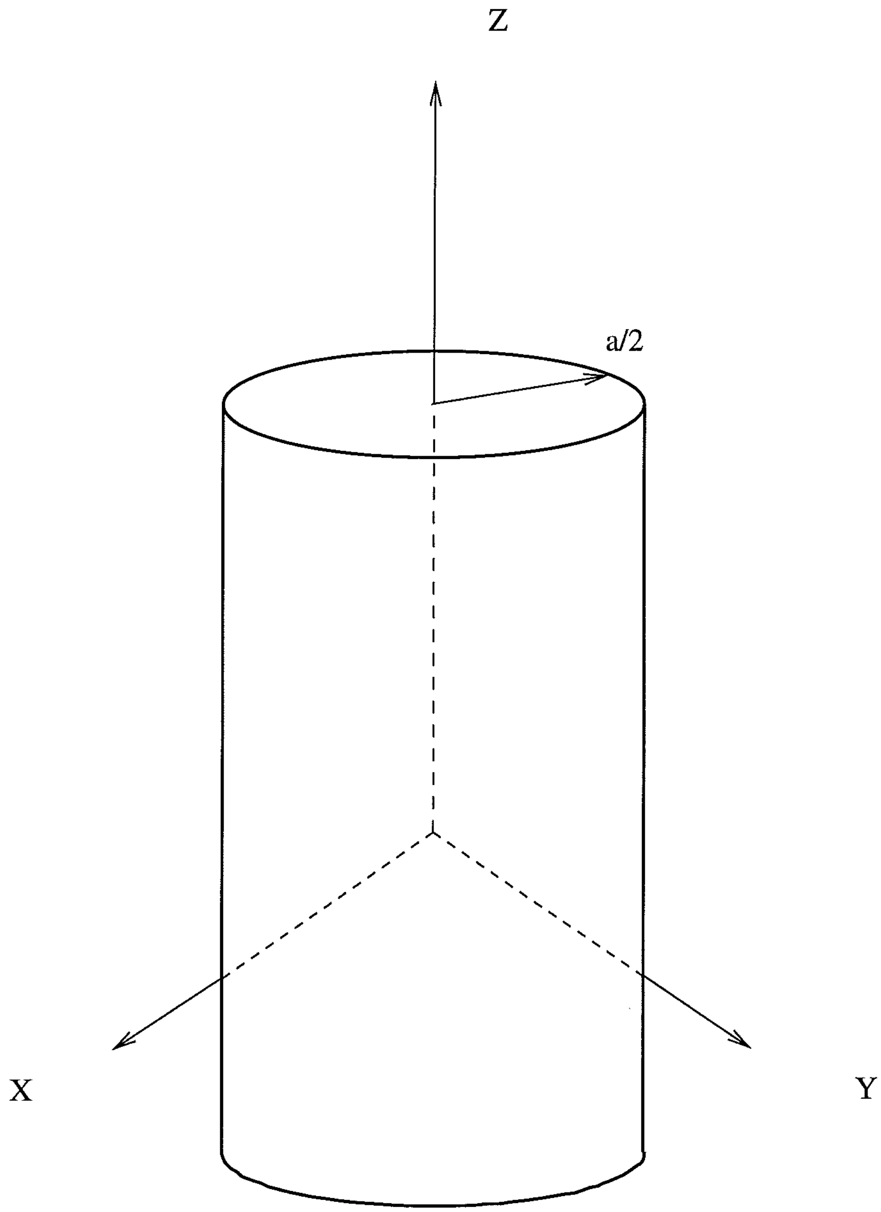

Figure 2. The Simplest Projected Constraint Manifold

CASE STUDY

In this section we present an example of the design of a planar RR dyad for 10 position rigid body guidance. The 10 desired positions are listed in Tbl. 1. These are the same 10 positions that were used by Ravani and Roth (1983) to demonstrate their numerical constraint manifold synthesis procedure.

Using *VISSYN* in approximately 5 minutes of user time a dyad with $\mathbf{u} = [13.98 \;\; -2.53]^T$, $\mathbf{v} = [7.64 \;\; -8.26]^T$, $a = 6.87$, and $\psi = 157.34$ was found. This dyad is illustrated in Fig. 4 with the solid red circle being the fixed pivot, the thin green circle being the path of the moving pivot, and the solid green circles locate the moving pivot, using the vector $\boldsymbol{\lambda} = [10.23 \;\; -4.68]^T$, with respect to M. Note that if the dyad was an exact solution that the centers of the solid green circles would lie on the thin green circle. The projected constraint manifold of the solution dyad and the 10 desired positions are shown in Fig. 3.

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Pos.	dx	d_y	θ
1	0.0	0.0	40.0
2	4.5	4.0	20.0
3	8.5	8.0	0.0
4	13.0	11.5	−30.0
5	13.0	12.5	−35.0
6	9.5	14.0	−35.0
7	5.0	13.5	−30.0
8	1.0	10.5	−15.0
9	−1.0	6.5	0.0
10	−1.5	3.0	20.0

Table 1.

In Ravani and Roth (1983) two solution dyads are presented. Using their metric the position errors of the dyads are 0.016 and 0.013 while the error of the dyad obtained using *VISSYN* is 0.008, as is listed in Fig. 4.

CONCLUSION

In this paper we have derived the constraint manifold of the planar *RR* dyad. The constraint manifold yields an analytical representation of the workspace of the dyad in the image space of planar displacements. We then presented a synthesis procedure for approximate rigid body guidance which utilizes the visualization of the projection of the constraint manifold onto $\Re^3$. The design process has been implemented into the program *VISSYN* and its use was illustrated in a design case study.

By employing modern computer graphics we have utilized the constraint manifold of the planar *RR* dyad to synthesize solutions for approximate rigid body guidance without encountering the numerical difficulties that are usually involved. It is our hope that by employing advanced synthesis tools such as kinematic mappings with a display of the physical design space will yield new and useful machines and mechanisms.

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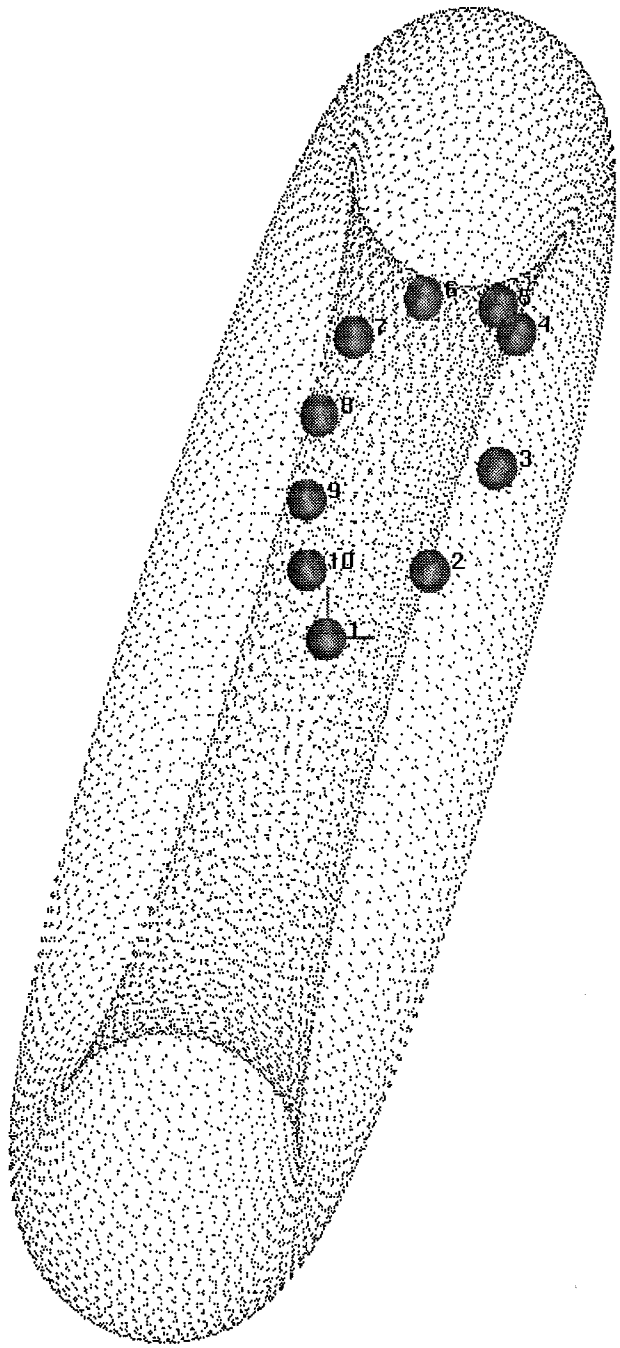

Figure 3. *VISSYN*

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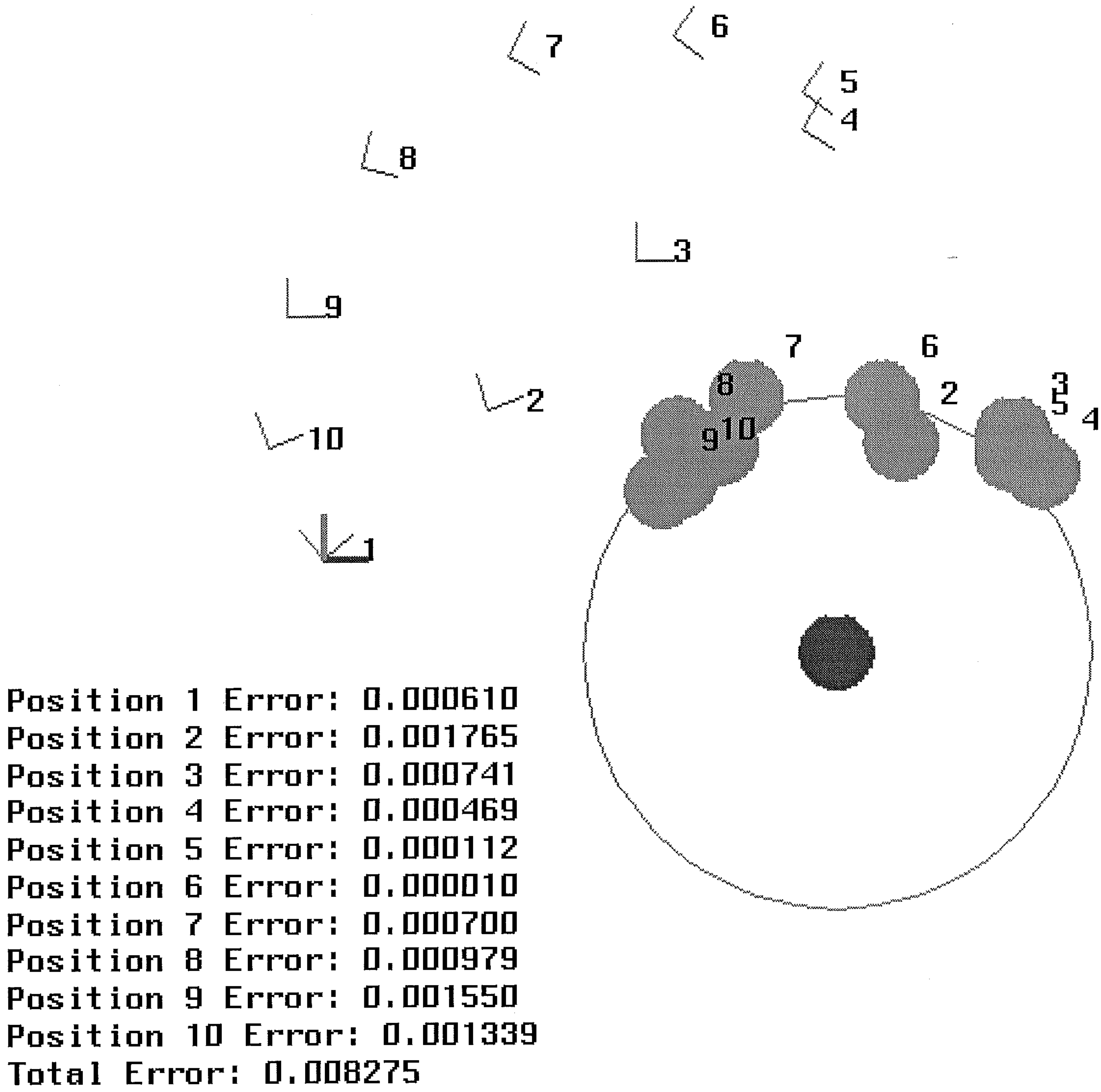

Figure 4. *VISSYN* **Design Space Display**

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