

# Approximating Spatial Locations With Spherical Orientations for Spherical Mechanism Design

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*In this paper we present a novel method for approximating a finite set of  $n$  spatial locations<sup>1</sup> with  $n$  spherical orientations. This is accomplished by determining a design sphere and the associated orientations on this design sphere which are nearest the  $n$  spatial locations. The design sphere and the orientations on it are optimized such that the sum of the distances between each spatial location and its approximating spherical orientation is minimized. The result is a design sphere and  $n$  spherical orientations which best approximate a set of  $n$  spatial locations. In addition, we include a modification to the method which enables the designer to require that one of the  $n$  desired spatial locations be exactly preserved. This method for approximating spatial locations with spherical orientations is directly applicable to the synthesis of spherical mechanisms for motion generation. Here we demonstrate the utility of the method for motion generation task specification in spherical mechanism design. [S1050-0472(00)00204-X]*

## Introduction

Spherical mechanisms are linkages which generate spherical motion of rigid bodies. In spherical motion, the displacement of any point on the body is constrained to the surface of a sphere. In contrast, planar mechanisms generate two-dimensional motion. For this reason their design is compatible with using conventional drafting tools while the synthesis of spherical mechanisms is three-dimensional and is not well suited for these drafting techniques. It is essential that the spherical mechanism designer be able to visualize the entire problem in three-dimensions and computer graphics can be an effective tool for providing this necessary visualization of the problem to the designer. Efforts have been made to create computer graphics based software packages for spherical four-bar mechanism design:

- SPHINX was the first spherical mechanism computer-aided design (CAD) program written by Larochele et al. [1] for use on Silicon Graphics workstations. SPHINX begins by displaying a **design sphere**. The design sphere defines the surface in space upon which the workpiece is to be moved. The relative displacements between the locations on the design sphere are purely rotational and are called **orientations**. Orientations are defined by their longitude, latitude, and roll angles [2]. In SPHINX orientations are displayed to the designer as coordinate frames on the surface of the design sphere, see Fig. 1. The current version of SPHINX has modules for performing synthesis for three or four orientation rigid body guidance. It is important to note that in SPHINX the design sphere is of arbitrary radius and its location in space is undefined.

- SPHINXPC [3] is a CAD program for personal computers which like SPHINX utilizes a design sphere with orientations displayed on the sphere's surface. With this software spherical mechanisms can be designed for four orientations. SPHINXPC also can be used to design planar mechanisms for four location rigid body guidance.

- SPHEREVR [4] is the first virtual reality (VR) based approach

to spherical mechanism design. This initial exploration of the use of VR for spherical mechanism design has led to the development of a 3rd generation of VR based spherical mechanism design software called ISIS, see Larochele, Vance, and McCarthy [5]. The program utilizes the compute engine of SPHINX1.2 and provides virtual objects in the design environment so that the design process takes place in a virtual representation of the physical workspace. This new approach to mechanism design has demonstrated a need for new and efficient means for specifying the design task in the actual physical workspace of the mechanism.

To synthesize a spherical mechanism, the designer must first define the task to be accomplished. Here we are concerned with task specification for moving a workpiece through a sequence of prescribed orientations in space. This task is referred to as *rigid-body guidance* by Suh and Radcliffe [6] and as *motion generation* by Erdman and Sandor [7]. An example of a rigid body guidance task is shown in Fig. 2. The desired locations of the workpiece are defined in space. A coordinate frame is attached to the workpiece and each of its desired locations is recorded. To date, when designing spherical mechanisms the designer must determine an appropriate design sphere, i.e. its center and radius, from the desired

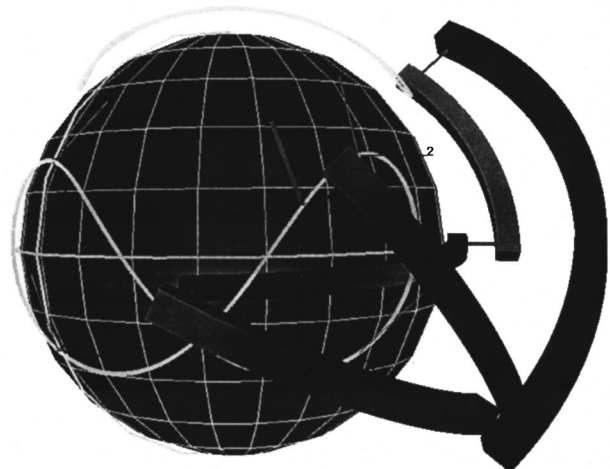


Fig. 1 SPHINX DESIGN SPHERE

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<sup>1</sup>The location or pose of a rigid body is defined by both its orientation and position; the position is defined by three coordinates which uniquely define where a point of the body is and the orientation is defined by three angles which orient the body with respect to a fixed reference body.

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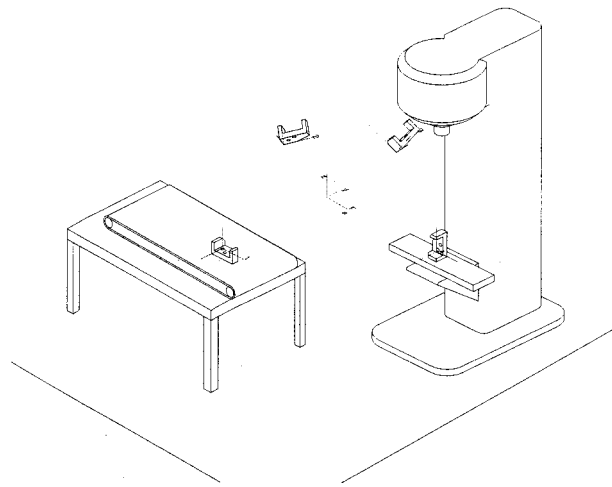


Fig. 2 A desired task

spatial locations. Moreover, the sets of angles which define the orientations of the body with respect to that design sphere must also be determined. Currently, no methodologies exist to facilitate this process. It is only after determining the design sphere and the orientations that the designer can utilize CAD tools such as SPHINX and SPHINXPC.

In this paper, one method of determining the optimal design sphere and orientations from a desired set of spatial locations is presented. First, the spatial locations are approximated with orientations in four-dimensional Euclidean space ( $\mathbf{E}^4$ ). Biquaternions are then used to represent these orientations. Next, the distance between the spatial locations and the orientations on a candidate design sphere are calculated using a bi-invariant metric on biquaternions. Finally, an optimization method is used to minimize the distances between the spherical orientations on the candidate design sphere and the spatial locations. The result is a procedure which numerically determines the optimal design sphere and orientations for a finite set of desired spatial locations.

### Orientations in $\mathbf{E}^4$ and Biquaternions

In [2] Larochelle and McCarthy presented an algorithm for approximating a set of  $n$  locations in planar Euclidean space ( $\mathbf{E}^2$ ) with  $n$  spherical orientations in three-dimensional Euclidean space ( $\mathbf{E}^3$ ). By utilizing a bi-invariant metric on the image space of spherical displacements they arrived at an approximate bi-invariant metric for planar locations in which the error induced by the spherical approximation is of the order  $1/R^2$ , where  $R$  is the radius of the approximating sphere. In this paper we extend their methodology to the general spatial case and utilize the results to provide a novel method of specifying motion generation tasks for spherical mechanisms.

It was shown in Larochelle and McCarthy [2] that orientations in  $\mathbf{E}^3$  may be used to approximate locations in a bounded region of a two-dimensional plane. We utilize the contributions of Etzel and McCarthy [8] and extend that idea by using orientations in  $\mathbf{E}^4$  to approximate locations in a bounded region of three-dimensional space. This can be done by using a small portion of a four-dimensional hypersphere, a *wedge*, to approximate a bounded region of space. Orientations on the surface of this wedge, which we represent with biquaternions, can be used to approximate the spatial locations. See Ge [9] in which he examines the theory of biquaternions as representations of orientations on a hypersphere.

We proceed by briefly reviewing quaternions and biquaternions. Recall that an orientation in  $\mathbf{E}^3$  can be represented by a quaternion  $\mathbf{q}=[q_1 \ q_2 \ q_3 \ q_4]^T$ . The four components of the quaternion  $\mathbf{q}$  (sometimes referred to as Euler parameters) are,

$$\begin{aligned} q_1 &= s_x \sin \frac{\theta}{2} = s_x s \frac{\theta}{2} \\ q_2 &= s_y \sin \frac{\theta}{2} = s_y s \frac{\theta}{2} \\ q_3 &= s_z \sin \frac{\theta}{2} = s_z s \frac{\theta}{2} \\ q_4 &= \cos \frac{\theta}{2} = c \frac{\theta}{2} \end{aligned} \quad (1)$$

where  $\mathbf{s}$  and  $\theta$  are the rotation axis and the angle of rotation associated with the orientation, respectively. Note that the components of  $\mathbf{q}$  satisfy the following constraint equation,

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 - 1 = 0 \quad (2)$$

and lie on a unit hypersphere which we denote as *the image space of spherical displacements*, see Larochelle [10] and McCarthy [11].

Recall that the location of a body in  $\mathbf{E}^3$  has six degrees of freedom (three to define orientation and three to define location) and can be represented by a  $4 \times 4$  homogeneous transform [12]:

$$T = \begin{bmatrix} [R(\theta, \phi, \psi)] & \vdots & \mathbf{d} \\ \dots\dots\dots\dots\dots\dots & & \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix} \quad (3)$$

$$[R(\theta, \phi, \psi)] = \text{Rot}_y(\theta) \text{Rot}_x(-\phi) \text{Rot}_z(\psi)$$

where  $\mathbf{d}$  is a  $3 \times 1$  translation vector. The angles  $\theta$ ,  $\phi$ , and  $\psi$  are the longitude, latitude, and roll angles respectively [2]. In 1996 Etzel and McCarthy [8] showed that a  $4 \times 4$  homogeneous transform in  $\mathbf{E}^3$  can be approximated by a pure rotation in  $\mathbf{E}^4$ ,

$$[D] = [J(\alpha, \beta, \gamma)][K(\theta, \phi, \psi)] \quad (4)$$

where,

$$J(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha & 0 & 0 & s\alpha \\ -s\beta s\alpha & c\beta & 0 & s\beta c\alpha \\ -s\gamma c\beta s\alpha & -s\gamma s\beta & c\gamma & s\gamma c\beta c\alpha \\ -c\gamma c\beta s\alpha & -s\beta c\gamma & -s\gamma & c\gamma c\beta c\alpha \end{bmatrix}$$

and,

$$K(\theta, \phi, \psi) = \begin{bmatrix} \vdots & 0 \\ [R(\theta, \phi, \psi)] & \vdots & 0 \\ \vdots & 0 \\ \dots\dots\dots\dots\dots\dots & \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

The angles  $\alpha$ ,  $\beta$  and  $\gamma$  are defined as follows:  $\tan(\alpha) = d_x/R$ ,  $\tan(\beta) = d_y/R$ , and  $\tan(\gamma) = d_z/R$  where  $d_x$ ,  $d_y$ , and  $d_z$  are the components of  $\mathbf{d}$  and  $R$  is the radius of the hypersphere.

The bounded spatial workspace must represent only a small portion of the hypersphere (i.e. a wedge), hence we determine the radius of the hypersphere as:

$$R = \frac{4L}{\epsilon^{1/2}} \quad (5)$$

where  $L$  is the largest component of the translation vectors from the set of spatial locations and  $\epsilon$  is the maximum allowable error in the approximation of the spatial locations with the orientations in  $\mathbf{E}^4$ . The result is that the  $n$  spatial locations lie within a  $2L$  cube and the wedge approximates a  $4L$  cube, with the center of each cube being the origin of  $\mathbf{E}^3$ . It is important to note that the selection of  $R$  determines the metric's weighting between rotational



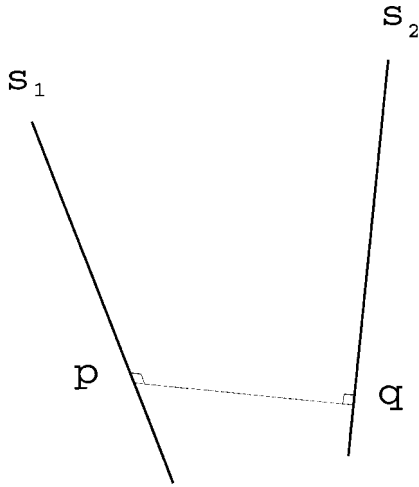


Fig. 4 Common normal of two screw axes

Minimize:

$$f(\mathbf{r}, \mathbf{c})$$

Subject to:

$$\|\mathbf{r}\| \leq 2L$$

$$\|\mathbf{c}\| \leq 2L$$

where:

$$f(\mathbf{r}, \mathbf{c}) = \sum_{i=1}^n d(\hat{Q}_i, \hat{R}_i)$$

and  $\hat{Q}_i$  and  $\hat{R}_i$  are the biquaternion representations of the  $n$   $T_{spherical}$  and  $T_{spatial}$  respectively. Note that the magnitudes of both  $\mathbf{r}$  and  $\mathbf{c}$  are bounded to insure that the design sphere remains within the  $4L$  cube of  $\mathbf{E}^3$  that is being approximated by the hypersphere's wedge, see Eq. 5.

We utilize the *simplex method for function minimization* to find  $\mathbf{r}$  and  $\mathbf{c}$  that minimize  $f(\mathbf{r}, \mathbf{c})$ , see Nelder and Mead [20]. This method was selected since it does not require analytical gradients and it is a direct multidimensional minimization algorithm.

**Initialization.** If the  $n$  spatial locations are in fact spherical orientations then the center of the design sphere is located at the intersection of the relative screw axes associated with the locations. However, with general spatial locations these relative screw axes will not intersect, see Bottema and Roth [21]. Hence, we find the point nearest all of the relative screw axes and use it as the initial center of the optimal design sphere. In Fig. 4 the common normal associated with two relative screw axes is shown. The intersections of the common normal with the two screw axes are  $\mathbf{p}$  and  $\mathbf{q}$ . Note that if the screw axes do not intersect then the point in space nearest the screw axes is the midpoint of the segment  $\overline{\mathbf{p}\mathbf{q}}$ . The initial estimation of the center  $\mathbf{c}$  is selected as the point nearest all of the relative screw axes associated with the spatial locations:

$$\mathbf{c}_{initial} = \frac{\sum_{i=1}^l \mathbf{p} + \sum_{i=1}^l \mathbf{q}}{2l} \quad (8)$$

where  $l = \binom{m}{2}$  and  $m = \binom{2}{2}$  is the number of relative screw axes<sup>4</sup>.

<sup>4</sup>Note that  $\binom{n}{k}$  denotes the binomial coefficient, often referred to as "n choose r."

The initialization of  $\mathbf{r}$  is obtained by equating the translation vectors of  $T_{spatial}$  and  $T_{spherical}$ . For any given spatial location the radial vector  $\mathbf{r}$  of the design sphere is then,

$$\mathbf{r} = [R]^T(\mathbf{d}_{spatial} - \mathbf{c}). \quad (9)$$

Substituting  $\mathbf{c}_{initial}$  into Eq. 9 we obtain:

$$\mathbf{r} = [R]^T(\mathbf{d}_{spatial} - \mathbf{c}_{initial}). \quad (10)$$

Using Eq. 10 we compute  $\mathbf{r}$  for each spatial location. The initial estimation of the radial vector is then the average,

$$\mathbf{r}_{initial} = \frac{\sum_{i=1}^n \mathbf{r}}{n}. \quad (11)$$

**Preserving One Position.** It may be necessary for the designer to require that one of the desired  $T_{spatial}$  be preserved. In this case the design sphere is constrained to exactly preserve this one spatial location (referred to as  $T_{exact}$ ). The design sphere is then optimized to minimize the distance between the remaining  $T_{spatial}$ 's and their associated  $T_{spherical}$ 's. Let us label the elements of the  $4 \times 4$  homogeneous transform representation of  $T_{exact}$  as,

$$T_{exact} = \begin{bmatrix} [R_{exact}] & \vdots & \mathbf{d}_{exact} \\ \dots\dots\dots & \dots & \dots\dots\dots \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}.$$

By equating the translation vectors of  $T_{exact}$  and  $T_{spherical}$  we obtain:

$$\mathbf{d}_{exact} = [R_{exact}]\mathbf{r} + \mathbf{c}. \quad (12)$$

We note that Eq. 12 is a linear system of three equations in the six unknown components of  $\mathbf{r}$  and  $\mathbf{c}$ . The simplex method for function minimization is employed to optimize the location of the center of the design sphere  $\mathbf{c}$  and Eq. 12 is used to determine  $\mathbf{r}$  at each iteration,

$$\mathbf{r} = [R_{exact}]^T(\mathbf{d}_{exact} - \mathbf{c}). \quad (13)$$

### Spherical Index

Obviously, not all finite sets of general spatial locations can be approximated with spherical orientations. Some sets of spatial locations are more near spherical than others and yield better spherical approximations while other sets of spatial locations may be far from spherical and for these no acceptable spherical approximations exist.

The method presented here does not guarantee an acceptable set of spherical orientations may be found for every set of general spatial locations. Recall that the purpose of this method is to facilitate the design of spherical mechanisms for motion generation. The implication being that the set of spatial locations will be *near spherical* and the method we present here determines the exact spherical orientations which best approximate the near spherical locations. As a measure of how near spherical the original spatial locations are we utilize the following **spherical index**  $\odot$ :

$$\odot = \frac{\sum_{i=1}^m |d_{relative}|}{4Lm} \quad (14)$$

where  $d_{relative}$  is the translation along the relative screw axes associated with two locations and  $m$  and  $L$  are as defined above. Sets of spatial locations with small  $\odot$  yield acceptable spherical approximations while sets with large  $\odot$  will not yield acceptable spherical approximations. It is important to note that the magnitude of  $\odot$  is dependent upon the choice of units used to define the spatial locations. Hence the spherical index  $\odot$  is only valid when used as a relative measure to compare sets of locations expressed with respect to the same units and within the same  $2L$  cube in



space. Furthermore, regardless of choice of units, a value of  $\odot = 0$  indicates that the  $n$  spatial locations are spherical and that an exact design sphere exists<sup>5</sup>.

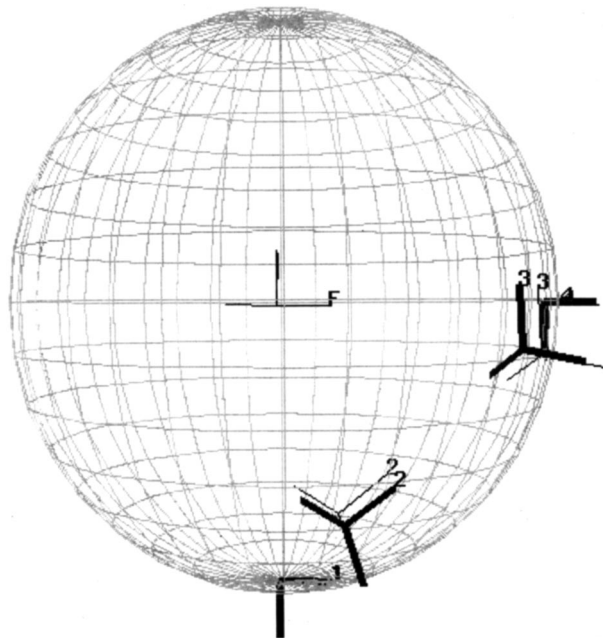
### Case Study: 1

We now illustrate the task specification methodology by applying it to the motion generation task shown in Fig. 2. The longitude, latitude, and roll angles (in degrees) and translation vectors for the four desired spatial locations are found in Table 1. The spherical index value for these locations is  $\odot = 7.211E-8$  which indicates that these locations are very near spherical. Hence, we anticipate that there exist spherical orientations which are very near the original spatial locations and proceed with the numerical nonlinear optimization. The initializations of the center and radial vectors are  $\mathbf{c}_{initial} = [0.2227 \ 0.2218 \ -0.1629]^T$  and  $\mathbf{r}_{initial} = [-0.1084 \ 0.2114 \ 5.1736]^T$ . The radius of the hypersphere is  $R = 2080$ , with  $\epsilon = 0.0001$  and  $L = 5.2$ . In Fig. 5 the optimal design sphere and orientations are shown. The spherical orientations are the coordinate frames with thicker lines. The optimal center and radial vectors for this design sphere are  $\mathbf{c} = [0.1019 \ 0.0791 \ 0.0244]^T$  and  $\mathbf{r} = [-0.0771 \ 0.0151 \ 5.0821]^T$ . The optimal orientations ( $1'$ ,  $2'$ ,  $3'$ ,  $4'$ ) and their distances from the original spatial locations are found in Table 1.

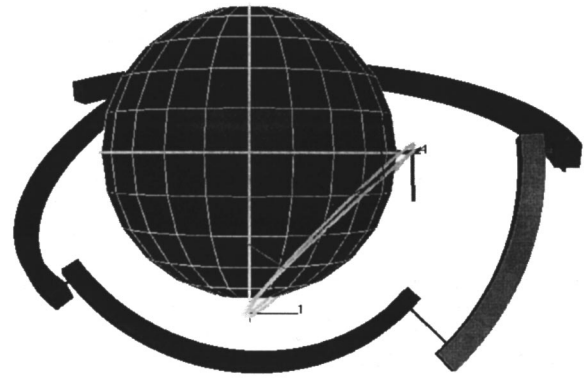
<sup>5</sup>Note that the design sphere will be exact and unique if  $\odot = 0$  and  $n > 3$  since the sphere passing through four or more points in space is unique.

**Table 1 Case 1: Desired spatial locations and their associated optimal orientations**

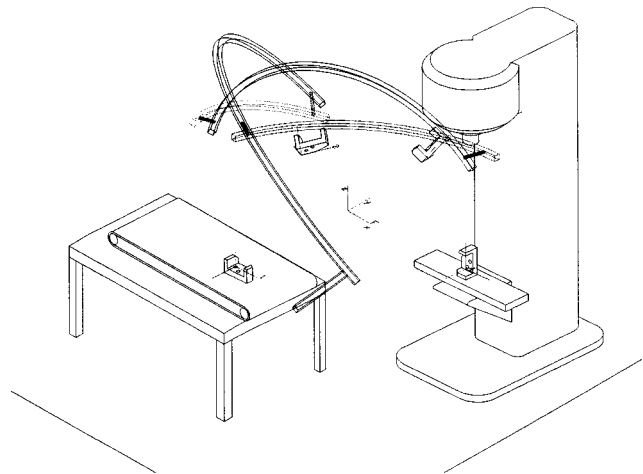
Pos.	Long.	Lat.	Roll	$d_x$	$d_y$	$d_z$	Distance
1	0.0	-90.0	0.0	0.00	-5.00	0.00	NA
1'	0.0	-90.0	0.0	0.02	-5.00	0.04	$1.555E-5$
2	14.12	-38.41	51.48	0.92	-2.98	3.65	NA
2'	14.12	-38.41	51.48	1.01	-3.12	3.87	$8.485E-6$
3	47.23	-7.46	108.55	4.00	-0.71	3.70	NA
3'	47.23	-7.46	108.55	3.80	-0.66	3.43	$1.555E-5$
4	90.0	0.0	180.0	5.20	0.00	0.00	NA
4'	90.0	0.0	180.0	5.18	0.06	-0.05	$1.697E-5$
TOTAL							$5.655E-5$



**Fig. 5 Case 1: Optimal design sphere and orientations for the desired task**



**Fig. 6 Case 1: A spherical mechanism for the desired task**



**Fig. 7 Case 1: A solution implementation for the desired task**

Having now determined the orientations which best approximate the original spatial locations we can now use SPHINX to design a spherical four-bar mechanism to generate the desired motion. The resulting mechanism, as displayed by SPHINX is shown in Fig. 6, and as implemented in the workspace is shown in Fig. 7. In order to employ this design to generate the desired motion manufacture the coupler for a radius of  $\|\mathbf{r}\|$ , manufacture the remaining links at appropriate radii, mount the mechanism such that the center of its associated sphere is located at  $\mathbf{c}$ , and attach the workpiece to the coupler.

### Case Study: 2

We now illustrate the task specification methodology by applying it to a motion generation task with 10 prescribed locations. The longitude, latitude, and roll angles (in degrees) and translation vectors for the ten desired spatial locations are found in Table 2. The spherical index value for these locations is  $\odot = 0.049$  which indicates that these locations are somewhat near spherical and perhaps an acceptable solution exists. Hence, we proceed with the numerical nonlinear optimization. The initializations of the center and radial vectors are  $\mathbf{c}_{initial} = [0.2227 \ 0.2218 \ -0.1629]^T$  and  $\mathbf{r}_{initial} = [-0.1084 \ 0.2114 \ 5.1736]^T$ . The radius of the hypersphere is  $R = 2103$ , with  $\epsilon = 0.0001$  and  $L = 5.26$ . The nonlinear optimization algorithm required 1837 iterations and run-time of  $\sim 0.1$  seconds on an R4400 SGI Indigo<sup>2</sup> to converge to the following solution. In Fig. 8 two views of the original locations (first row) and the optimal orientations (second row and thicker lines) are shown. The left view in each row is looking down the z-axis

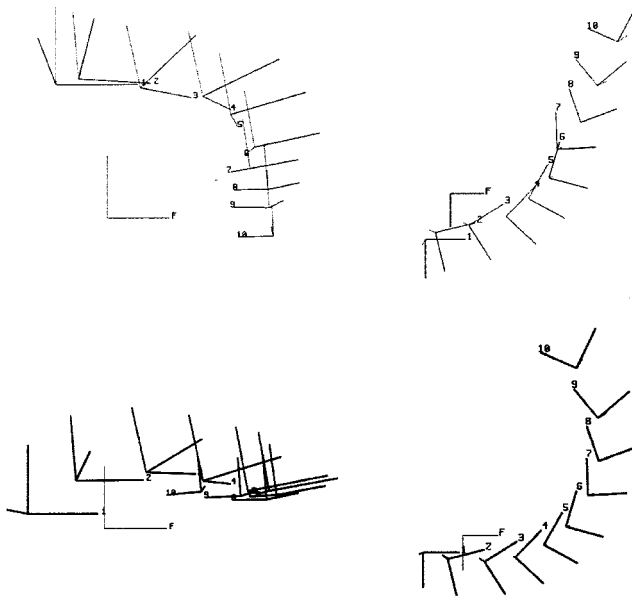
**Table 2 Case 2: Desired spatial locations and their associated optimal orientations**

Pos.	Long.	Lat.	Roll	$d_x$	$d_y$	$d_z$	Distance
1	0.0	0.0	0.0	-.6225	1.6500	1.1675	NA
1'	0.0	0.0	0.0	-1.0845	0.2150	.4727	.0006
2	-14.48	-9.33	-5.12	-.3645	1.8068	0.9737	NA
2'	-14.48	-9.33	-5.12	-.3854	0.6671	0.6254	.0004
3	-33.01	-8.86	-10.80	0.4514	1.76089	0.7980	NA
3'	-33.01	-8.86	-10.80	0.5763	0.7666	0.6935	.0003
4	-46.80	-4.85	-10.38	1.3665	1.7332	0.5735	NA
4'	-46.80	-4.85	-10.38	1.3951	0.6690	0.5801	0.0004
5	-61.14	-.83	-9.83	1.9438	1.6230	0.1262	NA
5'	-61.14	-.83	-9.83	2.1731	0.5663	0.2511	0.0004
6	-74.65	0.27	-10.19	2.5714	1.2402	-.4189	NA
6'	-74.65	0.27	-10.19	2.7480	0.6299	-.2390	.0003
7	87.49	4.53	-10.51	2.8586	0.9905	-1.2134	NA
7'	87.49	4.53	-10.51	3.3322	0.6505	-1.0799	.0002
8	70.25	7.46	-8.69	3.6236	0.6505	-2.0136	NA
8'	70.25	7.46	-8.69	3.6236	0.6505	-2.0136	0.0
9	50.24	10.56	-8.09	4.2561	0.2748	-3.1739	NA
9'	50.24	10.56	-8.09	3.5363	0.8341	-3.1255	0.0005
10	26.07	13.31	-9.68	5.2582	-.5780	-4.8453	NA
10'	26.07	13.31	-9.68	2.8865	1.0959	-4.3250	0.0034
TOTAL							0.0065

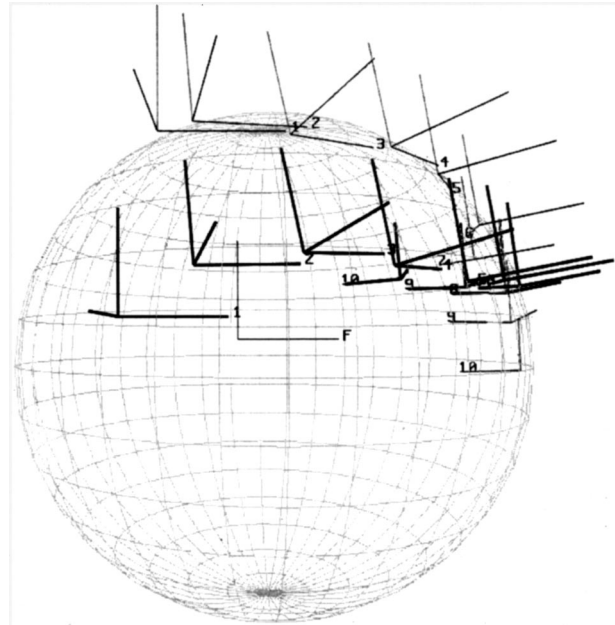
of the fixed frame while the right views are looking down the y-axis of the fixed frame. Moreover, in Fig. 9 the optimal design sphere and both the original locations and the optimal orientations are shown. The optimal center and radial vectors for this design sphere are  $\mathbf{c}=[0.4271 \ -0.3261 \ -2.4987]^T$  and  $\mathbf{r}=[-1.5112 \ 0.5411 \ 2.9713]^T$ . The optimal orientations (1', 2', 3', etc.) and their distances from the original spatial locations are found in Table 2. Note that the total error in the spherical approximations is 0.0065 and that the error at any one location is not large relative to the other location errors. In general, this indicates that an acceptable set of spherical orientations which approximate the original spatial locations has been found.

**Case Study: 3**

We now illustrate the task specification methodology by applying it to a motion generation task with 5 prescribed locations which are far from being spherical. The purpose of this case study



**Fig. 8 Case 2: Ten original locations and their optimal orientations**

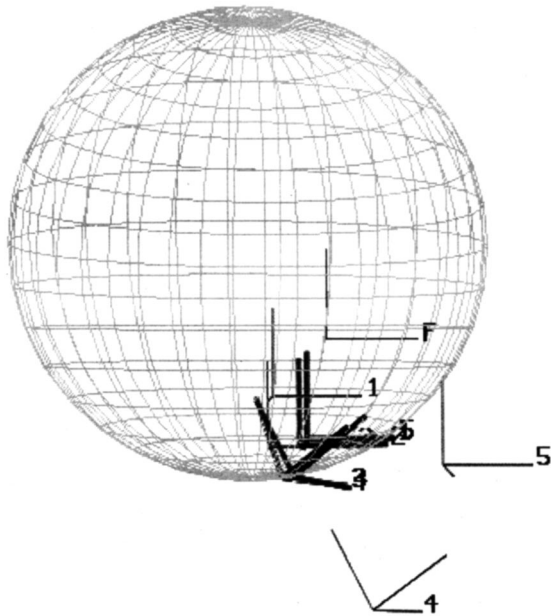


**Fig. 9 Case 2: Ten original locations and their optimal orientations**

is to discuss how the spherical approximation technique performs under such situations. Since the locations are intentionally far from being spherical we expect a large spherical index value and that all sets of spherical approximations determined by the nonlinear optimization will have large errors associated with them. The longitude, latitude, and roll angles (in degrees) and translation vectors for the desired spatial locations are found in Table 3. The spherical index value for these locations is  $\odot = 0.106$  which indicates that these locations are not near spherical. Nevertheless, we proceed with the numerical nonlinear optimization. The initializations of the center and radial vectors are  $\mathbf{c}_{initial}=[-59.7023 \ 64.7726 \ -14.5160]^T$  and  $\mathbf{r}_{initial}=[61.1154 \ -65.4455 \ -11.1851]^T$ . The radius of the hypersphere is  $R=1116$ , with  $\epsilon = 0.0001$  and  $L=2.79$ . The nonlinear optimization algorithm required 2965 iterations to converge to the following solution. The large number of iterations was required since the locations are not near spherical and that results in the initialization of the algorithm ( $\mathbf{c}_{initial}$  and  $\mathbf{r}_{initial}$ ) not being good initial estimates of the final solution. However, even in this case the run-time (~0.1 sec) of the algorithm is still acceptable. In Fig. 10 the original locations and the optimal orientations (with thicker lines) are shown. The optimal center and radial vectors for this design sphere are  $\mathbf{c}=[0.7704 \ 0.9344 \ 0.6147]^T$  and  $\mathbf{r}=[0.5474 \ -2.0626 \ -1.0550]^T$ . Note that they vary greatly from their initial estimates. The optimal orientations (1', 2', 3', etc.) and their distances from the original spatial locations are found in Table 3.

**Table 3 Case 3: Desired spatial locations and their associated optimal orientations**

Pos.	Long.	Lat.	Roll	$d_x$	$d_y$	$d_z$	Distance
1	-.60	-.60	-.03	-.6010	-.642	-.2620	NA
1'	-.60	-.60	-.03	-.2339	-1.1390	-.4292	0.0004
2	0.66	1.50	-.54	-.6710	-1.240	-.4340	NA
2'	0.66	1.50	-.54	-.3087	-1.209	-.3130	0.0002
3	-54.31	-31.36	-20.74	-.4540	-1.4190	0.7830	NA
3'	-54.31	-31.36	-20.74	-.4539	-1.4189	0.7830	0.0
4	-66.98	47.13	-29.54	0.4570	-2.790	-.7040	NA
4'	-66.98	47.13	-29.54	-.3280	-1.3800	0.9540	.0016
5	0.00	0.00	0.00	1.275	-1.415	-.2225	NA
5'	0.00	0.00	0.00	-.2220	-1.1280	-.4403	.0010
TOTAL							0.0033



**Fig. 10 Case 3: Five original locations and their optimal orientations**

Note that the total error in the spherical approximations is surprisingly small (0.0033) and this indicates that perhaps an acceptable set of spherical orientations which approximate the five original spatial locations has been found. Upon further examination of Fig. 10 and Table 3 it is evident that the spherical approximations are not very close to some of the original locations. This is a typical result for sets of locations which are not spherical. Often, a subset of the prescribed locations will be very near spherical. The spherical approximation to these subsets have small location errors and hence are very strong local minima. Here, the optimal orientations for locations 4 and 5 have associated with them large errors since locations 1, 2, and 3 are very near spherical<sup>6</sup>. In this case the designer will have to determine how important locations 4 and 5 are to the desired task. If locations 4 and 5 were chosen to guide the moving body in some general direction (e.g., around an obstacle) then perhaps the optimal orientations are acceptable. However, if either location four or five is critical to the task at hand then the optimal orientations most likely are not acceptable.

## Summary

In this paper we have presented a novel method for approximating a finite set of spatial locations with orientations on a design sphere. This was accomplished with a new methodology for determining the optimal design sphere and the orientations on this design sphere for a finite set of desired spatial locations. Moreover, we have included a modification to the algorithm such that one of the desired spatial locations is exactly preserved. The result

<sup>6</sup>In fact, the spherical index for the first three locations is  $\odot = 1.762E-7$ .

is that mechanism designers can now specify spherical mechanism motion generation tasks without having to introduce into the design space an artificial design sphere.

Finally, we believe that the utility of this new task specification algorithm will be most evident when utilized in three-dimensional computer graphics design environments such as SPHINXPC and SPHINX. Moreover, we anticipate that it will be an asset to the new Isis virtual reality spherical mechanism design environment currently being created in a collaborative effort led by Prof. J. M. Vance at Iowa State University and Prof. P. M. Larochelle at the Florida Institute of Technology.

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