DETC2000/MECH-14053

CIRCUIT AND BRANCH RECTIFICATION
OF THE SPATIAL 4C MECHANISM

Pierre M. Larochelle
Robotics and Spatial Systems Laboratory
Mechanical Engineering Program
Florida Institute of Technology
Melbourne, Florida 32904
Email: pierrel@fit.edu

ABSTRACT

Spatial 4C mechanisms are two degree of freedom kinematic closed-chains consisting of four rigid links simply connected in series by cylindrical(C) joints. In this work we are concerned with the design of spatial 4C mechanisms which move a rigid body through a finite sequence of prescribed locations in space. This task is referred to as *rigid-body guidance* by Suh and Radcliffe (20) and as *motion generation* by Erdman and Sandor (6). When 4C mechanisms are synthesized for such a task, for example by utilizing Roth's spatial generalization of Burmester's planar methods (17; 18), the result is the physical dimensions which kinematically define the mechanism. However, the motion of the mechanism which takes the workpiece through the sequence of prescribed locations in space is not determined. In fact, it may be impossible for the mechanism to move the body through all of the desired locations without disassembling the mechanism. This condition is referred to as a circuit defect. Moreover, in some cases the mechanism may enter a configuration which requires an additional mechanical input to guide the moving body as desired. These are referred to as branch defects. This paper presents a methodology for analyzing spatial 4C mechanisms to eliminate circuit and branch defects in motion generation tasks.

INTRODUCTION

Once the kinematic synthesis of a 4C mechanism has been completed (20; 18; 19; 7; 13; 3; 15; 8; 9; 1) the four lines in space which define the joint axes of the mechanism are determined, see Fig. 1. The synthesis guarantees that the mechanism may be assembled in each of the prescribed spatial locations. However, the continuous motion which takes the workpiece from one location to another is not determined and in fact it may not exist. The mechanism may suffer from circuit and/or branch defects (2; 16) which make it impossible for the desired motion to be realized by one assembly of the mechanism with one driving link.

Our goal here is to identify and eliminate those spatial 4C mechanisms which suffer from circuit or branch defects [1]. We study the motion of the spatial 4C mechanism which results from relative rotation and/or translation of the input or driving link with respect to a fixed link. Motion which results from rotating the input link while the link's relative translation is constant is referred to as rotational or angular motion of the mechanism. Similarly, motion which results from translating the input link while its relative rotation is constant is referred to as translational motion of the mechanism. First, we examine the angular motion of the spatial 4C mechanism for circuit or branch defects. This will be followed by a discussion of circuit and branch defects which may occur with respect to the translational motion of the mechanism.

CIRCUIT AND BRANCH DEFECTS

A **circuit** is defined as "all possible orientations of the links which can be realized without disconnecting any of the joints"

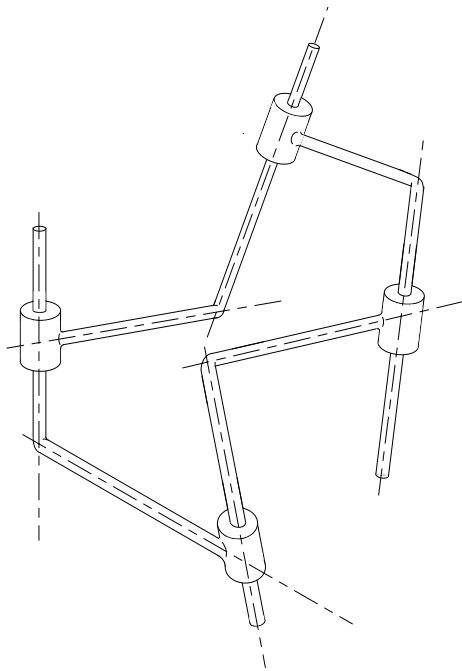[1]We utilize the terminology and definitions of Chase and Mirth (2).

Figure 1. A SPATIAL 4*C* MECHANISM.

(2). Obviously, those spatial 4*C* mechanisms which must be disconnected in order to guide the moving body through all of the desired locations must be rejected.

A **branch** is defined as "a continuous series of positions of the mechanism on the circuit between two stationary configurations" where stationary configurations are "positions of the linkage where the derivative of the angle of the output link with respect to the angle of the driving link becomes infinite" (2). Simply put, branches are the continuous series of positions between the stationary configurations on a circuit. Chase states that, "A mechanism that changes branch may suffer from drivability problems." In four-bar mechanisms, the drivability problem is due to the fact that when driving the input link through a singular configuration the motion of the output link is not determined. There exist rare instances in which useful mechanisms experience a change in branch in their useful range of motion(see (2)) however in general changes in branch are to be avoided.

Angular Motion

The circuit and branch tests presented here are based upon the works of: Reinholtz, Sandor, and Duffy (16); Chase and Mirth (2); and Mark (12). We identify circuit and branch defects associated with the angular motion of spatial 4*C* mechanisms by studying their associated spherical images. Associated with each spatial 4*C* mechanism is a spherical image. The spherical image is a spherical four-bar mechanism consisting of four links con-

nected by revolute joints; where the lengths of the links (α, η, β, and γ) are equal to the angular twists of the 4*C* mechanism, see Fig. 2 and (8; 5). Reinholtz, Sandor, and Duffy prove that "the spherical mechanism and the spatial mechanism must have the same branching characteristics"[2]. The branch test proposed by Reinholtz, Sandor, and Duffy(RSD) requires that the algebraic sign of the component along the fixed output axis of the cross product of a vector in the plane of the coupler and a vector in the plane of the output link must not change. Explicitly written for a set j of desired locations their branch test is a scalar triple product which simplifies to the following,

$$\forall j\ s_4 \times s_2^j \cdot s_3^j < 0\ \ \text{OR}\ \ \forall j\ s_4 \times s_2^j \cdot s_3^j > 0 \qquad (1)$$

where: s_4 is a unit vector along the fixed output axis, s_2 is a unit vector along the driving moving axis, and s_3 is a unit vector along the driven moving axis. The branch test is sufficient to also identify circuit defects for all four-bar mechanism types save the rocker-crank and double-rocker mechanisms. Both Chase and Mirth and Mark found that this test fails to prevent changes in circuit for these four-bar mechanisms. To address this deficiency Chase and Mirth propose that the range of the motion of the input link associated with each circuit be identified. For all desired locations of the moving body to be on one circuit requires that the input angle for each location lie within a single range. Mark simplifies and implements this idea by proposing two post checks to the RSD test:

Mechanism Type	RSD Post Check
rocker-crank	associated crank-rocker must pass the RSD test
double-rocker	$\forall j\ \theta^j > 0\ \ \text{OR}\ \ \forall j\ \theta^j < 0$

where θ^j is the relative input angle associated with location j. Spherical four-bar mechanisms which pass tests given by Eq. 1 and Eq. 2 will not suffer from circuit defects. Moreover, from Reinholtz, Sandor, and Duffy we have that any spatial four-bar mechanism associated with a spherical four-bar that passes these tests will not suffer from circuit defects with regard to its angular motion. Branch defects associated with the angular motion of the 4*C* mechanism will be dealt with in the next section.

Translational Motion

Cylindrical joints permit relative rotation and/or relative translation along the line of the joint. Circuit defects associated with the rotational or angular motion have been discussed above.

---

[2]The use of the term branch by Reinholtz, Sandor, and Duffy corresponds to Chase's definition of branch. Moreover, in their work they were addressing the *RCCC* mechanism but their claims are equally valid for the angular motion of the 4*C* mechanism.
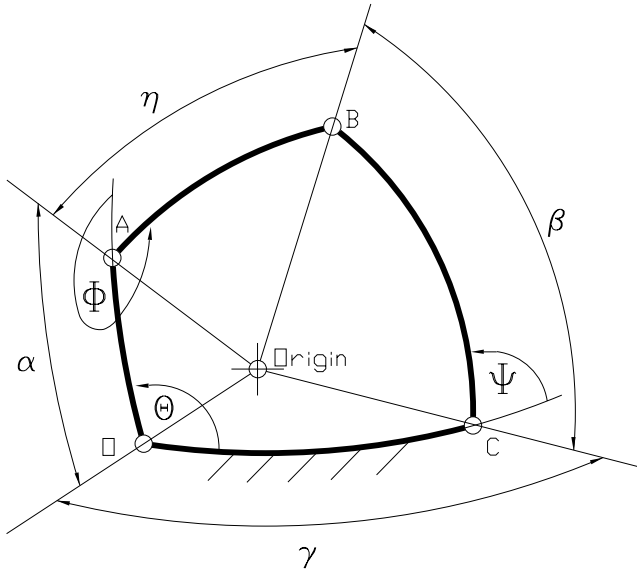
Figure 2. A SPHERICAL 4*R* MECHANISM.

We proceed here in a similar fashion with regard to circuits and branches associated with the translational motion of spatial 4*C* mechanisms.

Branch Analysis. In order to study conditions under which branching occurs with respect to the translational motion of the spatial 4*C* mechanism we propose the following definition of translational singular configurations:

Proposition 0.1. *Translational singular configurations are positions of the linkage where the translation of the output link with respect to the angle of the driving link becomes infinite.*

Marble and Pennock (11) show that the translational singular configurations of spatial 4*C* mechanisms occur at "angular locking positions" of the input link[3]. Obviously, infinite translations at the joints are practical impossibilities and will lead to catastrophic failure of the mechanism. Therefore, we eliminate all spatial 4*C* mechanisms with singular configurations. Four-bar mechanisms with fully rotatable driving links have two circuits associated with them "but neither contains singular configurations" (2). Hence, we avoid all branching defects by eliminating all spatial 4*C* mechanisms which do not have fully rotatable driving links[4].

---

[3]Marble and Pennock refer to Worle's definition of 'locking position' (21). Moreover, Worle's term locking position corresponds to Chase and Mirth's definition of singular configuration.

[4]There exist of course useful 4*C* mechanisms with rocker inputs as long as they are never permitted to near singular configurations. However, due to the catastrophic failure which results at any time when a 4*C* mechanism approaches

We utilize the results of Murray and Larochelle (14) to yield a simple check on the rotatability of the driving link of a spatial 4*C* mechanism. Let α be the angular twist of the driving link, β the angular twist of the driven link, η the coupler twist, and γ the twist of the fixed link. The driving link of the spatial 4*C* mechanism will be fully rotatable, i.e. a crank, if and only if $T_1 T_2 \geq 0$ and $T_3 T_4 \geq 0$; where $T_1 = \gamma - \alpha + \eta - \beta$, $T_2 = \gamma - \alpha - \eta + \beta$, $T_3 = -\gamma - \alpha + \eta + \beta$, and $T_4 = 2\pi - \gamma - \alpha - \eta - \beta$.

Circuit Analysis. We now proceed to address circuit defects associated with the translational motion of spatial 4*C* mechanisms. Consider the *CC* dyad shown in Fig. 3. The result of the kinematic synthesis for rigid body guidance is the specification of the fixed and moving lines of the dyad and the constraint that the link must impose upon the relative motion of the lines is that the dual angle between the two lines $\hat{\alpha} = \alpha + \varepsilon a$ must remain constant. Often in the literature and in practice the link used is a physical realization of the common normal of the two lines. However, any link which maintains the rigid body constraint that $\hat{\alpha}$ remain constant will suffice. This is analogous to the use of the line connecting two pivots in planar four-bar mechanisms. In practice the link can take on any shape but the distance between the pivots, i.e. the link length, must remain constant. Here, in the case of spatial *CC* dyads, the link can take on any shape as well but the dual angle $\hat{\alpha}$ between the lines must remain constant.

First, for simplicity, let us utilize the Denavit-Hartenberg (4) coordinate convention and consider the case of the link physically being the common normal to the two lines of the dyad as shown in Fig. 3. It is evident from the figure that the fixed joint translation *d* must not equal zero because the link would collide with the fixed cylindrical joint. Once the dyad has been assembled therefore the algebraic sign of *d* can not change. Note that the same is true for the translations at all of the *C* joints; once assembled the translation along any of the joint axes can not change algebraic sign. Hence, we state the following:

Proposition 0.2. *A spatial* 4*C* *mechanism whose physical links are the common normals of the joint axes will avoid a change in translational circuit if the algebraic sign of all relative joint translations remains constant throughout the desired motion.*

It is important to recall that the spatial 4*C* mechanism possesses two degrees of freedom and, when performing motion generation tasks, the path of the mechanism is not unique. Hence, research to determine feasible paths which avoid a change in circuit is ongoing.

Now, we consider the case of spatial 4*C* mechanisms with links of arbitrary geometry. Again, let us utilize the Denavit-Hartenberg coordinate convention and consider the link as shown

---

a singular configuration we adopt the conservative approach here and eliminate this possibility entirely.
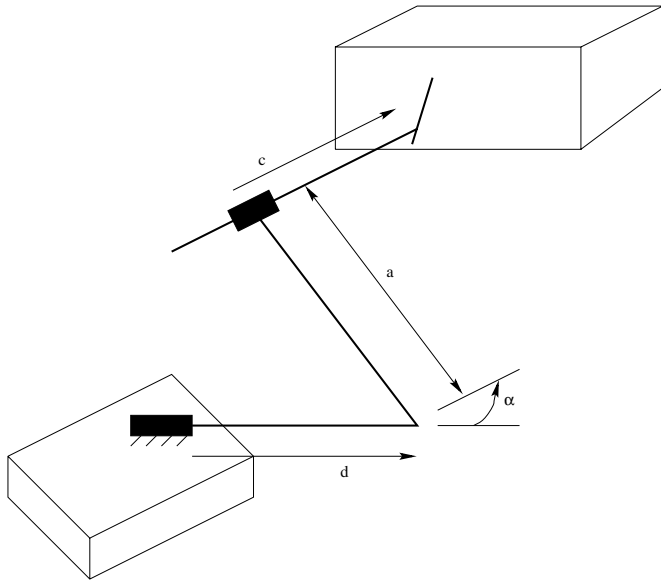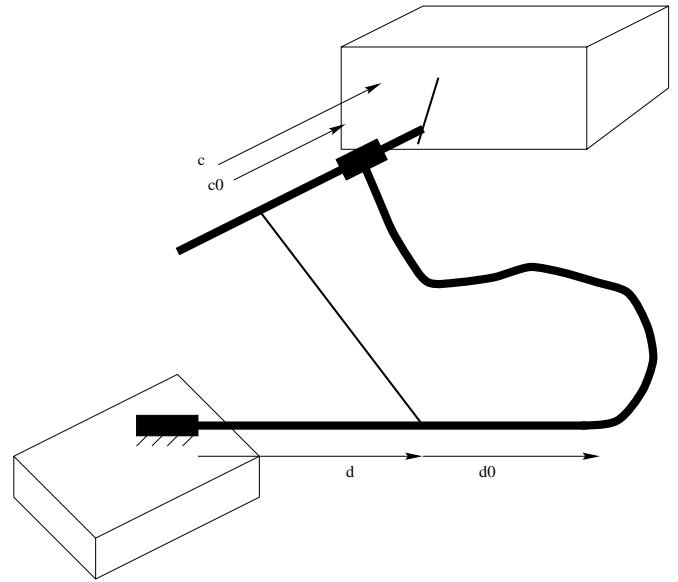
3

Figure 3.   A SIMPLE SPATIAL *CC* DYAD.

Figure 4.   A GENERAL SPATIAL *CC* DYAD.

in Fig. 4. Let us define d0 as the fixed axis offset; the signed distance from the intersection of the fixed axis and the link common normal to the link. Moreover, we define c0 as the moving axis offset; the signed distance from the intersection of the moving axis and the link common normal to the link. Following the same logic as in the simple dyad case, it is evident d must be greater than -d0 and c greater than c0 once the dyad has been assembled. Note that the same reasoning applies to each dyad of the mechanism. Hence, we state the following:

Proposition 0.3. *A spatial 4C mechanism whose physical links are of arbitrary geometry will avoid a change in translational circuit if throughout the motion both d>-d0 and c>c0 for each of its CC dyads.*

STEP-BY-STEP

Here we summarize the circuit and branch analysis procedure for identifying circuit and/or branch defects of spatial 4*C* mechanisms for rigid-body guidance tasks.

1. Is the input link fully rotatable?

   (a) From the angular link lengths determine T_1, T_2, T_3, and T_4.
   (b) Is $T_1 T_2 \geq 0$ and $T_3 T_4 \geq 0$?
         YES: Proceed to Step 2.
         NO: Reject Mechanism.

2. Is there an angular circuit defect?

   (a) For each desired location of the workpiece j determine s_2^j and s_3^j.
   (b) Evaluate the RSD test in Eq. 1.
         Test satisfied: Proceed to Step 3.
         Test failed: Reject Mechanism.

3. Is the mechanism physically constructed of simple links[5]?

   (a) YES:
         Perform a kinematic position analysis of the mechanism, e.g. see Larochelle (8).
         Do any of the joint translations experience a change of algebraic sign while guiding the moving body through the desired locations?
            YES: Reject Mechanism.
            NO: Done- no circuit or branch defects.

   (b) NO:
         Determine d0 and c0 for the driving *CC* dyad.
         Determine d0 and c0 for the driven *CC* dyad.
         Perform a kinematic position analysis of the mechanism, e.g. see Larochelle (8).
         For each dyad is d>-d0 and c>c0 throughout the motion?
            YES: Done- no circuit or branch defects.

---

[5]Simple links have geometries which essentially correspond to the common normal of their joint axes.

4

NO: Reject Mechanism.

CASE STUDY

We apply the circuit and branch analysis technique to an example for four desired locations[6] of the workpiece listed in Tbl. 1. A candidate solution mechanism is listed in Tbl. 2 and shown in Fig. 5. In Tbl. **??** the fixed joint axes are given with respect to the fixed frame while the moving joint axes are given with respect to the moving frame attached to the workpiece. Moreover, in Fig. 5 the driving link is green while the driven link is red and the moving body is represented by a coordinate frame(x-axis red, y-axis green, and z-axis blue) attached to the coupler. Application of the branch and circuit tests presented here yields:

1. $T_1T_2 = .682$ and $T_3T_4 = .005 \implies$ Driving link is fully rotatable.
2. $RSD^1 = -.12$, $RSD^2 = -.20$, $RSD^3 = -.34$, and $RSD^4 = -.33 \implies$ No angular circuit defect.
3. Mechanism consists of simple links and no joint translations experience a sign change during the desired motion$\implies$ No circuit or branch defects.

Table 1.   Desired Workpiece Locations

Pos. #	x	y	z	lng	lat	rol
1	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	1.0	0.25	15.0	15.0	0.0
3	1.0	2.0	0.5	45.0	60.0	0.0
4	2.0	3.0	1.0	45.0	80.0	0.0

CONCLUSIONS

In this work we addressed the issue of branch and circuit analysis of spatial $4C$ mechanisms for rigid-body guidance. Effort was made to adhere to the widely accepted terminology of Chase and Mirth(2). The techniques employed in the analysis are geometry based and build upon many recent contributions in the field. The result is a methodology for analyzing spatial $4C$ mechanisms to eliminate circuit and branch defects in motion generation tasks. A *step-by-step* summary of the analysis methodology and a numerical example were included. It is our hope that systematically addressing practical issues in spatial mechanism de-

---

[6]For definition of the longitude, latitude, and roll angle convention used here see Larochelle and McCarthy (10).

sign (e.g. branch and circuit defects)will enable the wide-spread application of spatial mechanisms in new products and processes.

ACKNOWLEDGMENT

REFERENCES

R. M. C. Bodduluri. Interactive graphics for four position synthesis of 4c spatial mechanisms. In *Proc. World Congress on the Theory of Machines and Mechanisms*, pages 81–84, Prague, Czechoslovakia, August 1991.

T.R. Chase and J.A. Mirth. Circuits and branches of single-degree-of-freedom planar linkages. *ASME J. of Mechanical Design*, 115:223–230, June 1993.

H.J. Cotner and A.G. Erdman. A "standard form" solution technique for spatial mechanism synthesis. In *Proc. ASME DETC Mechanisms Conf.*, pages 451–456, Kissimmee, Florida, September 1988.

J. Denavit and R. S. Hartenberg. A kinematic notation for lower pair mechanisms based on matrices. *ASME J. Appl. Mech*, 22:215–221, 1955.

J. Duffy. *Analysis of Mechanisms and Robotic Manipulators*. Wiley and Sons, New York, New York, 1980.

A. Erdman and Sandor G. N. *Mechanism Design: Analysis and Synthesis*, volume 1. Prentice Hall, Englewood Cliffs, New Jersey, third edition, 1997.

K. C. Gupta. Synthesis of position, path, and function generating 4-bar mechanisms with completely rotatable driving links. *Mechanism and Machine Theory*, 15:93–101, 1980.

P. M. Larochelle. S_{PADES}- software for synthesizing spatial 4c mechanisms. In *Proc. ASME DETC Mechanisms Conf.*, Atlanta, Georgia, August 1998.

P. M. Larochelle and J. M. McCarthy. Design of the spatial 4c mechanism for rigid body guidance. In *Proc. ASME DETC Mechanisms Conf.*, volume 70, pages 135–142, Minneapolis, Minnesota, September 1994a.

P. M. Larochelle and J. M. McCarthy. Planar motion synthesis using an approximate bi-invariant metric. *ASME J. of Mechanical Design*, 117(3):646–651, 1995.

S.D. Marble and G. R. Pennock. A study of the coupler curves of the rccc spatial four-bar mechanism. In *Proc. World Congress on the Theory of Machines and Mechanisms*, volume 1, pages 656–665, Oulu, Finland, June 1999.

D. C. Mark. Spherical linkage classification and singularity analysis. Master's thesis, University of California, Irvine, 1993.

J. M. McCarthy. The synthesis of planar rr and spatial cc chains and the equation of a triangle. *ASME J. of Mechanical Design*, 117(B):101–106, June 1995.

A. Murray and P. M. Larochelle. A classification scheme for

Table 2. Solution Mechanism Joint Axes

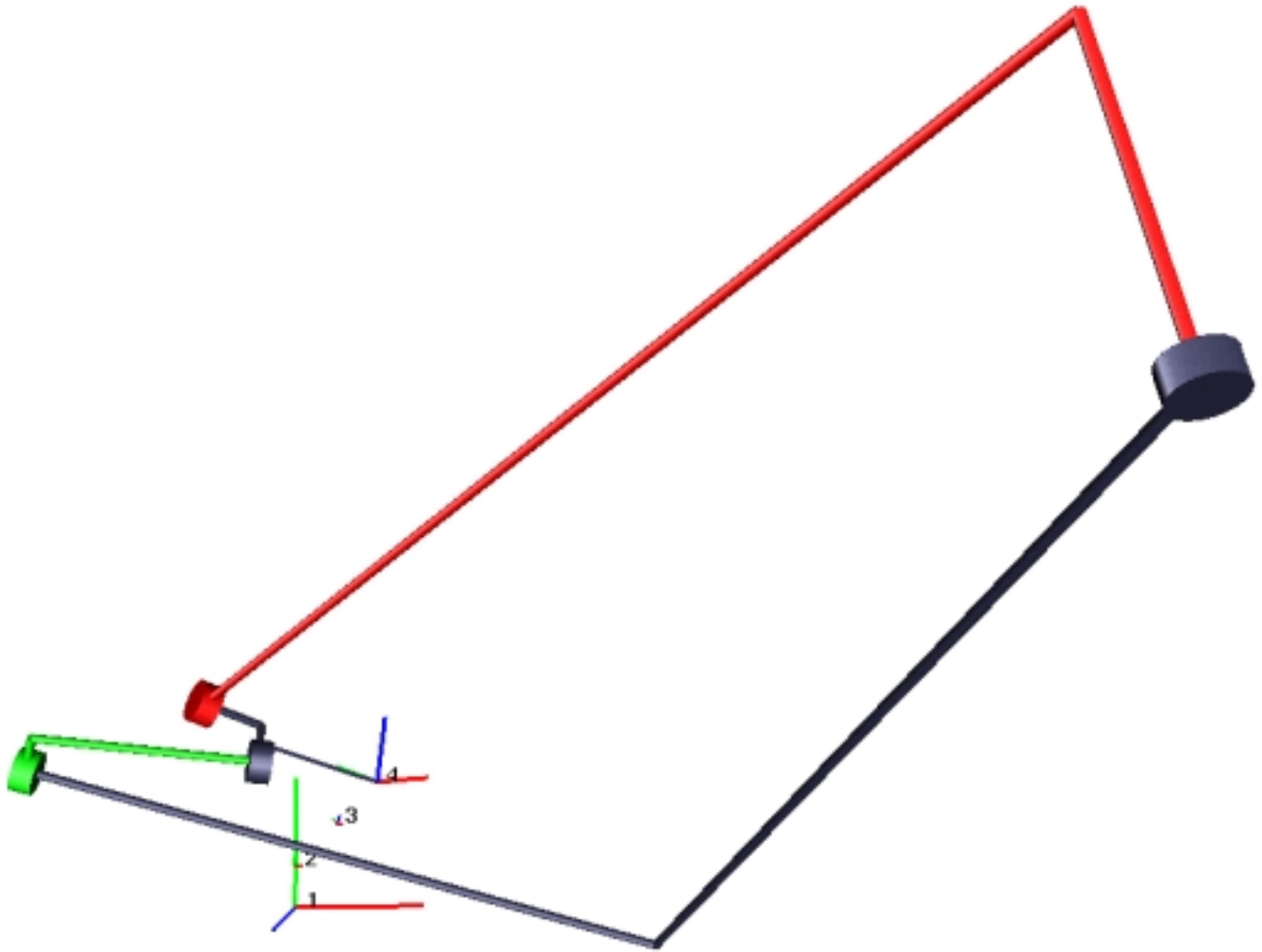| Joint Axis	Plücker Coordinates					
Driving Fixed	−0.954690	0.285803	0.082964	−0.157172	−0.971298	1.537413
Driving Moving	0.862913	−0.503524	−0.042949	−0.034063	0.015482	−0.865884
Driven Moving	−0.899995	0.290510	0.324983	0.706222	−0.260092	2.188287
Driven Fixed	−0.080628	0.900022	−0.428321	−22.594671	4.023614	12.707980

Figure 5. SOLUTION MECHANISM IN LOCATION 4.

planar 4r, spherical 4r, and spatial rccc linkages to facilitate computer animation. In *Proc. ASME DETC Mechanisms Conf.*, Atlanta, Georgia, August 1998.

B. Ravani and B. Roth. Motion synthesis using kinematics map-

pings. *ASME J. of Mech., Trans., and Auto. in Des.*, 105:460–467, September 1983.

C.F. Reinholtz, G.N. Sandor, and J. Duffy. Branching analysis of spherical rrrr and spatial rccc mechanisms. *ASME J.*

of Mech., Trans., and Auto. in Des., 108:481–486, December 1986.

B. Roth. The kinematics of motion through finitely separated positions. *ASME J. of Appl. Mech*, 34:591–598, 1967a.

B. Roth. Finite position theory applied to mechanism synthesis. *ASME J. of Appl. Mech*, 34:599–605, 1967b.

B. Roth. The design of binary cranks with revolute, cylindric, and prismatic joints. *J. of Mechanisms*, 3(2):61–72, 1968.

C. H. Suh and C. W. Radcliffe. *Kinematics and Mechanisms Design*. Wiley and Sons, New York, New York, 1978.

H. Worle. Untersuchungen uber koppelkurven viergliedriger raumlicher kurbelgetriebe. *Konstruction*, 14(10):390–392, 1962.