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Interactive Visualization of the Line Congruences Associated with Four Finite Spatial Poses

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Abstract

This paper presents a framework for generating and interacting with the line congruences associated with four general finite poses. These line congruences are the solution space of spatial 4C mechanisms which will guide a workpiece through the four prescribed poses. Hence, the contributions of this paper are applicable to developing interactive tools for designing spatial 4C mechanisms for four pose motion generation.

First, a methodology for generating a parameterized representation of the line congruences is reviewed. This is followed by strategies for visually representing the line congruences which are appropriate for both workstation and immersive virtual reality computer graphics. Next, strategies and supporting algorithms for interacting with the line congruences to obtain solution mechanisms with fixed links or coupler links in desired regions of the workspace are presented.

The result is an intuitive interactive visual design methodology for generating and interacting with the line congruences associated with four general finite spatial poses. Our hope is that this tool will in turn facilitate the creation of both virtual reality and workstation-based computer-aided design software for spatial 4C mechanism design.

1 Introduction

The purpose of this paper is to present a framework for visually representing and interacting with the line congruences associated with four spatial poses or locations. These line congruences are the result of the spatial generalization of the center point and circle point curves of planar kinematics; consequently they are the set of lines that define the axes of CC dyads that guide a rigid body through four prescribed locations in space[1]. Two CC dyads compatible with the four prescribed locations can then be connected in parallel to form a simple closed kinematic chain which is referred to as a spatial $4C$ mechanism. The resulting mechanism possesses two degrees of freedom and can be seen in Fig. 2.

In order to synthesize a spatial $4C$ mechanism to guide a body through four prescribed locations a designer may: (1) generate the fixed and moving congruences associated with the four locations; and subsequently, (2) select two lines from the congruences to define a $4C$ mechanism which is compatible with the four prescribed locations. We compute the congruences by employing the spatial triangle technique presented by Murray and McCarthy (1994). The result is a two dimensional parameterized set of lines. Our first utilization of this congruence generation technique resulted in the compute engine that powered the spatial $4C$ mechanism synthesis and analysis design software *Spades*, see Larochelle (1998). *Spades* successfully deployed the state of the art in spatial mechanism kinematic synthesis and analysis techniques on a traditional computer graphics workstation based platform. Though users have found *Spades* useful for designing mechanisms they have stated that one of the major challenges to designing spatial mechanisms for four desired locations is visualizing and interacting with the line congruences. This work focuses upon techniques for interactive visualization of the congruences to synthesize spatial $4C$ mechanisms for four location motion generation via the selection of CC dyads from the line congruences.

Our goal is to derive novel techniques for visually representing and interacting with the congruences to select CC dyads compatible with the four desired locations that possess desired properties such as axis location in the design space. These techniques are suited for both traditional workstation based computer graphics as well as immersive virtual reality environments such as the C6 and C2 facilities at Iowa State University. We hope that these techniques will in turn facilitate the creation of the next generation of computer-aided design software for $4C$ mechanism design for spatial motion generation.

[1] A cylindric(C) joint is a two degree of freedom lower pair joint which allows translation along and rotation about a line in space. Moreover, a CC dyad consists of a rigid link that is connected to ground via one C joint and connected to the moving body via a second C joint.

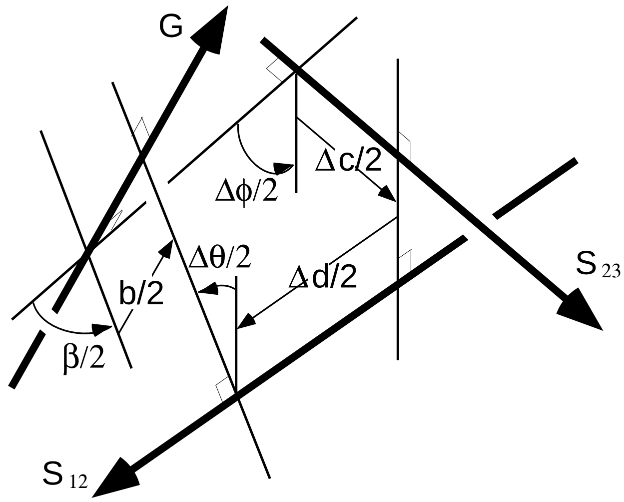

Figure 1: The Spatial Triangle

2 Congruence Generation

Before proceeding to the interactive visualization of the congruences it is instructive to review the congruence generation technique of Larochelle (1995). First, we review the spatial triangle technique of Murray and McCarthy (1994). This is followed by a methodology for utilizing the spatial triangle to generate the fixed and moving line congruences. Finally, a methodology for completing a dyad once one line has been selected from a congruence is presented.

2.1 The Spatial Triangle

In Murray and McCarthy (1994) it is shown that the spatial triangle prescribed by the finite relative screw axes S_{23} and S_{12} with internal dual angles $\frac{\Delta\hat{\phi}}{2}$ and $\frac{\Delta\hat{\theta}}{2}$ defines the coordinates of a fixed line G of a CC dyad compatible with four spatial locations, see Fig. 1. The dual vector equation of the spatial triangle may be written as,

$$\begin{aligned} \sin\frac{\hat{\beta}}{2}G &= \sin\frac{\Delta\hat{\theta}}{2}\cos\frac{\Delta\hat{\phi}}{2}S_{12} + \\ &\quad \cos\frac{\Delta\hat{\theta}}{2}\sin\frac{\Delta\hat{\phi}}{2}S_{23} + \\ &\quad \sin\frac{\Delta\hat{\theta}}{2}\sin\frac{\Delta\hat{\phi}}{2}S_{12}\times S_{23} \end{aligned} \tag{1}$$

$$\cos\frac{\hat{\beta}}{2} = \cos\frac{\Delta\hat{\theta}}{2}\cos\frac{\Delta\hat{\phi}}{2} +$$

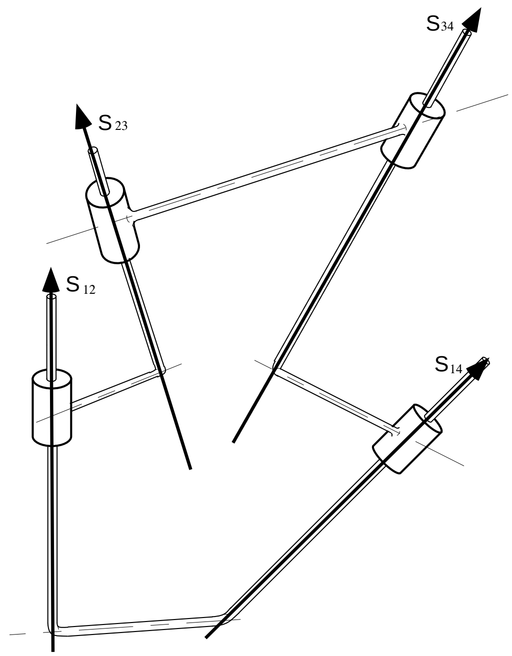

Figure 2: The Parameterizing Spatial $4C$ Mechanism

$$\sin\frac{\Delta\hat{\theta}}{2}\sin\frac{\Delta\hat{\phi}}{2}S_{12}\cdot S_{23} \quad (2)$$

In order to solve Eq. 1 and Eq. 2 for the desired line G the relationships between the spatial triangle, the complementary screw quadrilateral, and the $4C$ mechanism corresponding to the complementary screw quadrilateral must be maintained. We review those relationships here and outline the procedure for determining the line G given four spatial locations.

The generalization of Burmester's planar four location theory to four spatial displacements leads us to consider the complementary screw quadrilateral $S_{12}S_{23}S_{34}S_{14}$, where S_{12}, S_{13}, S_{14}, S_{23}, S_{24}, and S_{34} are the six relative finite screw axes associated with the four prescribed spatial locations, see Roth (1967b,1967c) and Bottema and Roth (1979). McCarthy (1993a) shows that by using the lines which define the complementary screw quadrilateral to define the axes of a spatial $4C$ mechanism one may obtain the fixed axis congruence in a parameterized form[2], see Fig. 2. The procedure presented in McCarthy (1993a) involves using the lines which define the complementary screw quadrilateral to define a spatial $4C$ mechanism and identifying the quadrilateral as the *home* configuration of the parameterizing $4C$ mechanism. This results in a $4C$ mechanism with input link defined by the lines $S_{12}S_{23}$, fixed link defined by the lines $S_{12}S_{14}$ and coupler defined by the lines $S_{23}S_{34}$. We define

[2]A planar version of this result, which yields a parameterized form of the center point curve for four planar locations, is found in McCarthy (1993b).

the input angle $\hat{\theta}_0$ as the dual angle of the input link in the home configuration and similarly define the coupler angle $\hat{\phi}_0$ as the dual angle between the coupler and the input link in the home configuration. The screw axis of the displacement of the coupler of the parameterizing $4C$ mechanism from its home configuration to any other valid assembly is a fixed axis compatible with the given four general spatial locations. Hence, we obtain fixed axes that are parameterized by the input angle $\hat{\theta}$ of the parameterizing linkage.

In their paper Murray and McCarthy (1994) show that solving the spatial triangle associated with the two lines which define the input link in its home configuration, Eq. 1 and Eq. 2, results in the relative screw axis of the displacement of the coupler of the parameterizing $4C$ mechanism, where $\Delta\hat{\theta} = \hat{\theta} - \hat{\theta}_0$ and $\Delta\hat{\phi} = \hat{\phi} - \hat{\phi}_0$. Therefore, by solving the spatial triangle we obtain a fixed axis compatible with the given four general spatial locations which is parameterized by the input angle of the parameterizing $4C$ linkage. Note that the internal angles of the spatial triangle are given in terms of the relative input and coupler angles of the parameterizing $4C$ linkage with respect to its home configuration.

3 The Fixed Congruence

We now present a method of obtaining a numerical representation of the fixed congruence which is parameterized by the input angle of the parameterizing $4C$ linkage using the spatial triangle. Recall that the fixed congruence is a two dimensional set of lines that define the fixed axes that are compatible with four spatial locations and that a solution of the spatial triangle presented by Murray and McCarthy (1994) yields one line of the fixed congruence. Bottema and Roth (1979) and Roth (1967a) have shown that the direction of each line G determines a unique plane and that all of the lines in that plane that are parallel to G are members of the fixed congruence. Hence, each line G defines a unique direction and corresponding to this direction there is an infinite set of compatible fixed lines of CC dyads.

We proceed with a method for using the spatial triangle to determine another line of the congruence, G_2, which is parallel to $G_1 = G$. These two lines then define the plane associated with G_1. By examining Eq. 1 we see that the direction of G_1 is independent of the translation along, and the location of, the axes of the parameterizing $4C$ mechanism. In other words, the direction of G_1 is solely dependent upon the directions of the axes of the parameterizing linkage, this result was first presented by Roth (1967a). Therefore, to obtain G_2 with the same direction as G_1 we maintain θ and vary our choice of d, where d is the translation of the input link of the parameterizing linkage along S_{12}, ($\hat{\theta} = \theta + \epsilon d$), and solve Eq. 1 and Eq. 2. Hence, for a given choice of parameter θ we select two different values of d which yield two lines G_1 and G_2. These two lines then define a plane of the congruence and any line in this plane parallel to G is a member of the congruence.

We can now parameterize the lines in the plane associated with G that are members of the fixed congruence in terms of λ, where λ is the distance of the line from G. Given,

$$G(\theta) = G_1(\theta) = \begin{bmatrix} \mathbf{g} \\ \mathbf{g}_1^0 \end{bmatrix} \qquad (3)$$

and,

$$G_2(\theta) \quad = \quad \begin{bmatrix} \mathbf{g} \\ \mathbf{g}_2^0 \end{bmatrix} \qquad (4)$$

the lines $L_G(\theta, \lambda)$ that lie in the plane defined by G_1 and G_2 and are parallel to G may be expressed as,

$$L_G(\theta, \lambda) \quad = \quad \begin{bmatrix} \mathbf{g} \\ (\mathbf{p}_g + \lambda \mathbf{n}) \times \mathbf{g} \end{bmatrix} \qquad (5)$$

where,

$$\mathbf{n} \quad = \quad \frac{\mathbf{g} \times (\mathbf{g}_2^0 - \mathbf{g}_1^0)}{\|\mathbf{g} \times (\mathbf{g}_2^0 - \mathbf{g}_1^0)\|} \qquad (6)$$

and,

$$\mathbf{p}_g \quad = \quad \mathbf{g} \times \mathbf{g}_1^0 \qquad (7)$$

Note that $\mathbf{n}$ is a unit vector in the direction of the common normal to the lines G_1 and G_2, that $\mathbf{p}_g$ is a point on G, and selecting $\lambda = 0$ in Eq. 5 yields the line G. In Eq. 5 we have the two dimensional set of lines of the fixed congruence associated with four spatial locations parameterized by the angle θ of the input link of the parameterizing $4C$ mechanism (which selects a plane of the congruence) and a distance parameter λ (which selects a line in that plane).

Having reviewed the relationships between the relative screw axes, the complementary screw quadrilateral, the parameterizing $4C$ mechanism, and the spatial triangle, we now summarize the procedure for determining the fixed line congruence given four spatial locations.

1. *From the four specified locations determine: the four relative screw axes* $(S_{12}, S_{23}, S_{34}, S_{14})$*, the link lengths of the corresponding parameterizing* $4C$ *linkage* ($\hat{\alpha} = S_{12} \cdot S_{23}$, $\hat{\eta} = S_{23} \cdot S_{34}$, $\hat{\beta} = S_{14} \cdot S_{34}$, $\hat{\gamma} = S_{12} \cdot S_{14}$)*, and the angles* $\hat{\theta}_0$ *and* $\hat{\phi}_0$*.*

2. *Select the parameter value* $\hat{\theta}$ *and compute the corresponding* $\hat{\phi}$ *by performing a kinematic analysis of the parameterizing* $4C$ *linkage*[3]*, see Larochelle (1998).*

3. *Compute the internal angles of the spatial triangle,* $\Delta\hat{\theta}$ *and* $\Delta\hat{\phi}$*, and solve the two dual vector triangle equations, Eq. 1 and Eq. 2, for the unknown line* $G_1(\theta) = G$*.*

[3] In general for each θ there are two solutions for ϕ; simply let θ vary from 0 to 4π and use the first solution for $0 \leq \theta < 2\pi$ and the second solution for $2\pi \leq \theta < 4\pi$.

4. *For the same parameter value θ select a new value for d and compute the corresponding $\hat{\phi}$ by performing a kinematic analysis of the parameterizing 4C linkage.*

5. *Compute the internal angles of the spatial triangle, $\Delta\hat{\theta}$ and $\Delta\hat{\phi}$, and solve the two dual vector triangle equations, Eq. 1 and Eq. 2, for the unknown line $G_2(\theta) = G$.*

6. *If $\theta < 4\pi$ then increment θ and go to step 2 else done.*

4 The Moving Congruence

The moving congruence is the two dimensional set of lines that define the moving lines of the CC dyads that are compatible with four spatial locations of a rigid body. We obtain a parameterized representation of the moving congruence by inverting the relationship between the fixed and moving coordinate frames and proceeding in an analogous manner to the generation of the fixed congruence. The inverted locations yield the relative screw axes S'_{12}, S'_{13}, S'_{14}, S'_{23}, S'_{24}. We then form a complementary screw quadrilateral, its corresponding parameterizing $4C$ mechanism, and solve the spatial triangle for a given choice of θ to obtain the lines $H = H_1$ and H_2 which define a plane of the moving congruence. Proceeding as we did in the generation of the fixed congruence we obtain the lines of the moving congruence associated with the parameter θ,

$$L_H(\theta, \mu) = \begin{bmatrix} \mathbf{h} \\ (\mathbf{p}_h + \mu\mathbf{n}) \times \mathbf{h} \end{bmatrix} \tag{8}$$

where,

$$\mathbf{n} = \frac{\mathbf{h} \times (\mathbf{h}_2^0 - \mathbf{h}_1^0)}{\|\mathbf{h} \times (\mathbf{h}_2^0 - \mathbf{h}_1^0)\|} \tag{9}$$

and,

$$\mathbf{p}_h = \mathbf{h} \times \mathbf{h}_1^0 \tag{10}$$

Again we note that $\mathbf{p}_h$ is a point on H and that selecting $\mu = 0$ in Eq. 8 yields the line H. The result, Eq. 8, is a two dimensional set of lines, given with respect to the moving frame, associated with four spatial locations that are parameterized by the angle θ of the input link of the parameterizing $4C$ mechanism (which selects a plane of the moving congruence) and a distance parameter μ (which selects a line in that plane).

4.1 Fixed & Moving Congruences

There is a one-to-one correspondence between lines of the fixed congruence and lines of the moving congruence. That is to say, selecting a line from the fixed congruence as the fixed axis of a CC dyad uniquely determines the corresponding moving axis, and vice verse, see Roth (1967a). Hence, selecting a fixed and moving line from the congruences to specify a CC dyad involves two free parameters; θ and either λ or μ. Therefore, to uniquely determine a $4C$ mechanism from the congruences requires the selection of four free parameters; (θ_1, λ_1 or μ_1) which define one dyad and (θ_2, λ_2 or μ_2) which define the second dyad. In the next section we discuss how to obtain the unknown line of a spatial CC dyad corresponding to a choice of θ and either λ or μ.

4.2 Completing the Dyad

Having selected a line from either the fixed or moving congruences it is now desired to determine the corresponding axis of the CC dyad which guides a body through the four spatial locations. It is well known that for three spatial locations of a rigid body there is a one-to-one correspondence between the moving and fixed axes of a CC dyad. Therefore, we employ traditional dyadic dimensional synthesis techniques for three spatial locations to obtain the unknown corresponding line. We now present a method of determining the unknown fixed line of a dyad once θ and μ have been selected. Note that the procedure may be kinematically inverted to obtain the moving line given a choice of θ and λ.

For a given choice of θ and μ from Eq. 8 we have the line L_H. Since there is one-to-one correspondence between the moving and fixed axes of a CC dyad for three spatial locations we select any three of the four prescribed locations and solve for the unknown fixed axis, L_G. Moreover, because the moving and fixed congruences are both parameterized by the angle θ, and having computed the fixed congruence, we know the direction $\mathbf{g}$ of the line L_G. Hence, we need only to determine the moment which locates the line L_G in the plane of the fixed congruence associated with the parameter θ. We write the rigid link constraint equation of the CC dyad for each of the three locations and arrive at the following system of equations which determine the moment of the unknown fixed line, see Larochelle (1994),

$$[P]\mathbf{g}_0 \quad = \quad \mathbf{b} \tag{11}$$

where,

$$[P] \quad = \quad \begin{bmatrix} (\mathbf{l}_2 - \mathbf{l}_1)^T \\ (\mathbf{l}_3 - \mathbf{l}_1)^T \\ \mathbf{g}^T \end{bmatrix} \tag{12}$$

and,

$$\mathbf{b} = \begin{bmatrix} -(\mathbf{l}_2^0 - \mathbf{l}_1^0)\cdot\mathbf{g} \\ -(\mathbf{l}_3^0 - \mathbf{l}_1^0)\cdot\mathbf{g} \\ 0 \end{bmatrix} \tag{13}$$

and finally,

$$L_i = \begin{bmatrix} \mathbf{l}_i \\ \mathbf{l}_i^0 \end{bmatrix} \tag{14}$$

are the coordinates of the moving line L_H in the i^{th} location. Eq. 11 can be solved to obtain the unknown moment $\mathbf{g}_0$. Thereby determining the fixed line L_G corresponding to a choice of moving line L_H as,

$$L_G = \begin{bmatrix} \mathbf{g} \\ \mathbf{g}_0 \end{bmatrix} \tag{15}$$

4.3 Numerical Considerations

Bottema and Roth (1979) have shown that the congruences associated with four spatial locations are $(9,3)$ congruences; nine lines(either real or imaginary) of the congruence pass through a general point and three lines(either real or imaginary) of the congruence lie in a general plane. Moreover, they have shown that at least one real line of the congruence passes through a general point. Theoretically, there are an infinite number of valid angles θ of the parameterizing linkage which yield an infinite number of planes of the congruence. In our formulation we generate a plane of the congruence for each value of the parameter θ. Hence, by selecting a finite set of values of θ we do not generate the complete fixed and moving congruences and we do not know whether or not the planes we have generated will pass though a general point in space.

5 Interactive Visualization

Once a numerical representation of the congruences has been generated the challenge in creating an efficient design environment lies in providing a means of visually representing and interacting with the congruences. The first visual representation of the line congruences associated with four spatial locations known to the author may be found in Figure 26 of Bottema and Roth (1979). In this work Bottema and Roth visually illustrate a portion of the cubic cone associated with the orientations of the four spatial locations and in a separate figure adjacent to this they illustrate a portion of one plane of the congruence associated with a direction from the cone. A representation such as this which decouples the direction from the moment of the line leads to difficulties in visualizing the relationship of the congruence to the physical workspace of the resulting spatial mechanism. However, it is important to note that in this work Bottema and Roth created a representation to explain the concept of the congruence; not for interactive design. In 1991 Bodduluri presented the first computer graphics generated representation of the line congruences for spatial 4C mechanism design, see Figure 4 of his work. Bodduluri chose to represent the planes of the congruences by using two short parallel line segments. Upon close examination it is evident that identifying which two lines in the figure represent a given plane is difficult. Moreover, in his design environment Bodduluri restricts the designer to selecting lines from the displayed line segments. Obviously, this arbitrarily eliminates most of the solution space to the spatial motion generation problem at hand. Murray (1993) presents a methodology for generating the congruences associated with four finite spatial locations. He elected to represent the congruences "by creating a pair of orthogonal lines for each plane generated in the congruence. The longer of the two lines corresponds to the direction of the lines in the plane.". Note that Murray's goal was to generate the congruences; no attempt was made at utilizing the congruences for design.

In the works of Murray and McCarthy (1994 & 1999) the planes of the congruence associated with four spatial locations are represented by two dimensional wireframe rectangles whose shorter sides define the direction of lines in the plane. In Murray and McCarthy (1996) the planes of the fixed line congruence are represented again by these rectangles but in addition a line through the centroid of the rectangle in the direction associated with plane of the congruence is drawn. Again, it is important to note that Murray and McCarthy were not attempting to create a visual representation of the congruences suited to design. Their purpose was to visually represent the intersection of two congruences(1994 & 1999) and to visually demonstrate that the congruence generated passed through six prescribed lines.

In Larochelle (1995 & 1998) an attempt was made at generating and representing the congruences for spatial mechanism design. The previous attempts at visually representing the congruences were combined into a new solid model representation of the congruences. The planes of the congruences were represented by rectangular parallelepiped solids whose major axes were displayed and they defined the direction of the lines associated with each plane; see Fig. 3.

All previous visual representations of the congruences fail to address the following visualization challenge: How to visually represent the information contained in an infinite number of infinite planes in a manner that is useful for mechanism design? In other words, how to present to the designer a visual representation of the infinite number of axis directions available as well as the infinite number of lines associated with each direction? Moreover, it is essential that the information be presented to the designer in such a manner that the relative locations of the lines to the physical workspace of the mechanism is obvious. This is because commonly the mechanism designer seeks a solution $4C$ mechanism which has links in some finite region of the physical workspace. Here we present methodologies for visually representing and interacting with the congruences in a prescribed region of the physical workspace.

5.1 Desired Fixed Link Region

First, we consider the case in which the designer seeks a solution $4C$ mechanism which has a fixed link in a prescribed region of space. For simplicity, we will consider here the desired region of space to be defined by a bounding sphere. Recall that the fixed link is defined by two lines from the fixed line congruence. Hence, if portions of any two planes of the congruence lie inside the sphere than a line may be selected from each of those portions to define a fixed link which also lies inside the prescribed sphere. This strategy for determining fixed links from the congruences which lie within a desired region of the workspace may be implemented as follows:

1. *Generate the fixed line congruence.*
2. *The designer defines the sphere(center* $\mathbf{p}$ *& radius* r*) in space in which a fixed link is desired.*
3. *The minimum distance* d_{min} *from each plane of the congruence to the center of the sphere is determined (see appendix).*
4. *If* $d_{min} > r$ *than no portion of that plane lies within the sphere; hence, this plane may be eliminated. Proceed to the next plane.*
5. *If* $d_{min} < r$ *than a portion of that plane lies within the sphere; hence, this plane contains at least one fixed line within the sphere. Determine the lines in the plane that lie within the sphere*[4] *and display them as a rectangular parallelepiped solid to the designer. Proceed to the next plane.*
6. *The designer then selects two lines from the displayed solids to define the fixed link*[5]*.*

5.2 Desired Coupler Link Region

Next, we consider the case in which the designer seeks a solution $4C$ mechanism which has a coupler, or floating, link in a prescribed region of the moving body. Again for simplicity, we will consider here the desired region of the moving body space to be defined by a bounding sphere. Recall that the coupler link is defined by two lines from the moving line congruence. Hence, if portions of any two planes of the congruence lie inside the sphere than a line may be selected from each of those portions to define a coupler link which also lies inside the prescribed sphere. This strategy for determining coupler links from the congruences which lie within a desired region of the workspace may be implemented as follows:

[4] Here the intersection of the plane and the bounding sphere will be a circle whose center is the point $\mathbf{p}_{min}$ in the plane nearest the sphere center and the radius of this circle is $r_c = \sqrt{(r^2 - d_{min}^2)}$ (see appendix).

[5] Appropriate techniques for selecting lines with desired geometric properties may be found in Larochelle (1995).

Pos.	X	Y	Z	Long.	Lat.	Roll
1	0.00	0.00	0.00	00.0	00.0	00.0
2	0.00	1.00	0.25	15.0	15.0	00.0
3	1.00	2.00	0.50	45.0	60.0	00.0
4	2.00	3.00	1.0	45.0	80.0	00.0

Table 1: The 4 Prescribed Locations

1. *Generate the moving line congruence.*
2. *The designer defines the sphere(center & radius) in the moving body in which a coupler link is desired.*
3. *The minimum distance d_{min} from each plane of the congruence to the center of the sphere is determined (see appendix).*
4. *If $d_{min} > r$ than no portion of that plane lies within the sphere; hence, this plane may be eliminated. Proceed to the next plane.*
5. *If $d_{min} < r$ than a portion of that plane lies within the sphere; hence, this plane contains at least one moving line within the sphere. Determine the lines in the plane that lie within the sphere and display them as a rectangular parallelepiped solid to the designer. Proceed to the next plane.*
6. *The designer then selects two lines from the displayed solids to define the coupler link.*

6 Case Study

In this section we present an example of the design of a spatial $4C$ mechanism for four location rigid body guidance. The goal is to move a pallet off of a flexible assembly line into a convenient location to perform an assembly operation on the underside of the pallet and then to return the pallet to the assembly line. A moving coordinate frame was assigned to the pallet and the $4C$ mechanism is to attach itself to the pallet at the points; $[-1\ 0.25\ 0]^T$ and $[1\ 0.25\ 0]^T$. It is at these two points that holes are to be drilled in the pallet. These holes will serve as journal bearings for the moving C joints of the $4C$ mechanism. This application was suggested by Mark Senti and his associates at GSMA Systems Inc, Melbourne, FL. The four desired locations are prescribed by the (X, Y, Z) coordinates of the origin of the moving frame and the $(Longitude, Latitude, Roll)$[6] angles which describe the orientation of the moving frame with respect to the fixed reference frame, see Tbl. 1. The relative screw axes associated with these locations are listed Tbl. 2 and the link lengths of the associated parameterizing $4C$ mechanism are found in Tbl. 3. The

[6] See Larochelle (1994) for a definition of this angle convention.

Screw	Direction			Moment		
S_{12}	-0.704	0.704	0.093	1.327	1.217	0.840
S_{13}	-0.770	0.552	0.319	1.102	0.800	1.275
S_{14}	-0.841	0.415	0.348	1.300	0.753	2.192
S_{23}	-0.734	0.530	0.424	1.355	0.553	1.655
S_{24}	-0.802	0.376	0.463	1.812	0.573	2.672
S_{34}	-0.707	0.000	0.707	3.894	1.245	3.894

Table 2: The 6 Relative Screws

Link	Length (deg, distance)
DRIVING	$(21.65, 1.043)$
COUPLER	$(35.00, 0.587)$
DRIVEN	$(32.80, 1.395)$
FIXED	$(23.62, 1.214)$

Table 3: The Parameterizing $4C$ Mechanism

fixed and moving congruences, shown in Fig. 3, were generated for sequential values of θ beginning at $\theta = 0.0$ and stepping in increments of 0.1 radians; the fixed planes are mauve while the moving planes are yellow.

From the computed congruences we seek a $4C$ mechanism with a driving link which has a moving line that is near the point $\mathbf{p}_{dvg} = [-1\ 0.25\ 0]^T$ and a driven link with a moving line that passes through the point $\mathbf{p}_{dvn} = [1\ 0.25\ 0]^T$; both points are given with respect to the moving frame. First, we determine the driving dyad. For each plane of the moving congruence the distance d_{min} from the plane to the point $\mathbf{p}_{dvg}$ was computed. The plane corresponding to $\theta = 183.35(deg)$ was found to be nearest the point $\mathbf{p}_{dvg}$, $d_{min} = 0.032135$, and the line in this plane nearest to $\mathbf{p}_{dvg}$ is,

$$H_{dvg} = \begin{bmatrix} 0.79989 \\ -0.59803 \\ 0.05039 \\ 0.01387 \\ 0.54772 \\ 0.42987 \end{bmatrix} \tag{16}$$

The point on this line nearest to $\mathbf{p}_{dvg}$ is $\mathbf{p}_{min} = [-1.01924\ 0.22462\ 0.00426]^T$. The corresponding fixed line of the CC dyad, whose moment was found using three

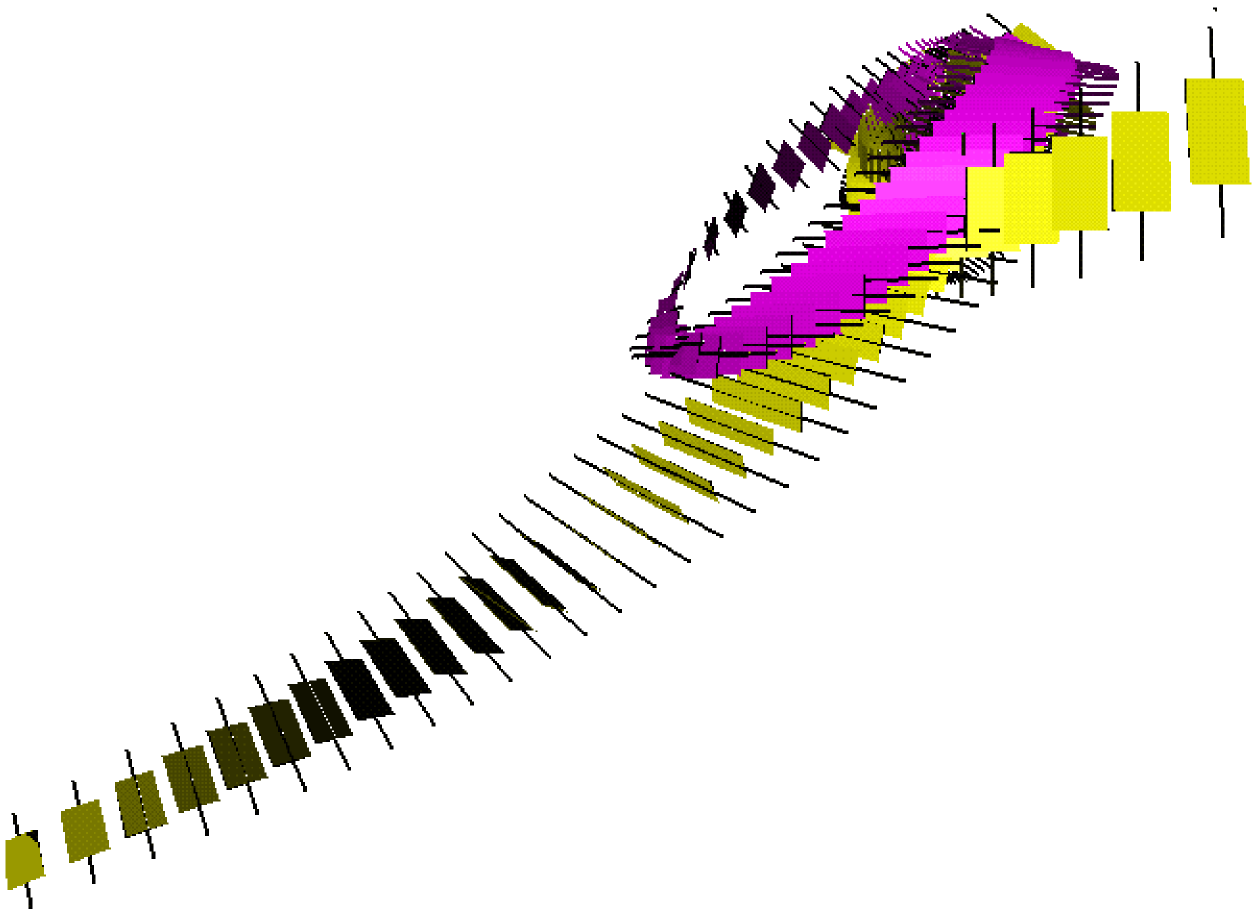

Figure 3: The Fixed and Moving Congruences

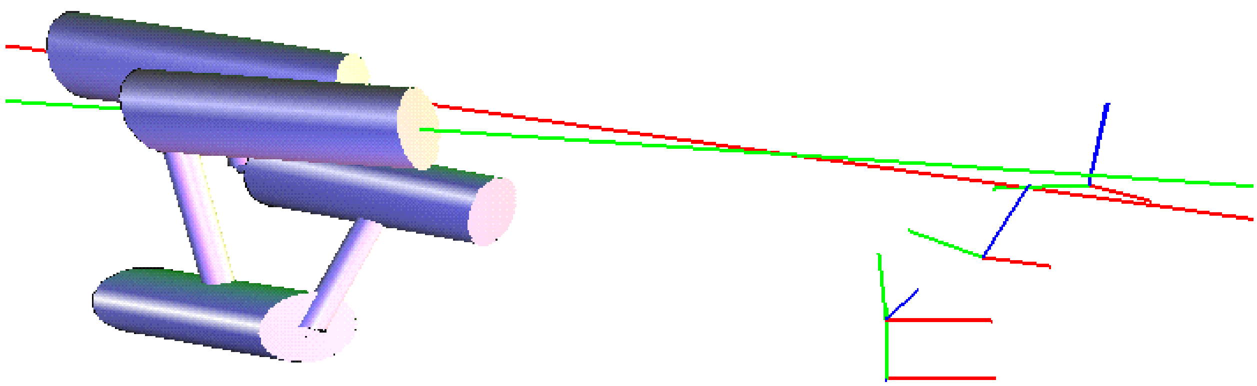

Figure 4: The $4C$ Mechanism

location synthesis, is,

$$G_{dvg} = \begin{bmatrix} 0.93653 \\ -0.23042 \\ 0.26423 \\ 0.59294 \\ 3.00646 \\ 0.52020 \end{bmatrix} \quad (17)$$

We obtain the driven dyad in an analogous manner. The plane corresponding to $\theta = 246.37(deg)$ was found to be nearest the point $\mathbf{p}_{dvn}$; $d_{min} = 0.019724$, and the line in this plane of the moving congruence nearest to $\mathbf{p}_{dvn}$ is,

$$H_{dvn} = \begin{bmatrix} 0.86004 \\ -0.50927 \\ -0.03126 \\ -0.01677 \\ 0.01674 \\ -0.73418 \end{bmatrix} \quad (18)$$

The point on this line nearest to $\mathbf{p}_{dvn}$ is $\mathbf{p}_{min} = [1.00459\ 0.25880\ -0.01704]^T$. The

Link	Length $(\mathrm{deg}, \mathrm{distance})$
DRIVING	$(25.81, 2.718)$
COUPLER	$(7.73, 0.668)$
DRIVEN	$(-165.41, 1.219)$
FIXED	$(-161.19, 2.112)$

Table 4: The Desired $4C$ Mechanism

corresponding fixed line of the driven dyad is,

$$G_{dvn} = \begin{bmatrix} -0.95907 \\ 0.27706 \\ 0.05847 \\ -0.25088 \\ -1.15251 \\ 1.34609 \end{bmatrix} \quad (19)$$

Finally, we show the resulting $4C$ mechanism in Fig. 4 and list its link lengths in Tbl. 4. In Fig. 4 the mechanism is shown with the moving body in location 4 and the lines which attach the moving body to the mechanism are also shown; where the driving moving line H_{dvg} is green while the driven moving line H_{dvn} is red.

7 Conclusion

In this paper we have presented a procedure for representing and interacting with the fixed and moving line congruences associated with four finitely separated spatial locations or poses. Moreover, algorithms for selecting lines from the congruences to define the axes of spatial $4C$ mechanism such that the fixed link or the coupler link lie within prescribed bounded regions of the workspace were included. The result is a methodology for performing the kinematic dimensional synthesis of spatial $4C$ mechanisms for four location rigid body guidance. The design process was illustrated in a detailed example. It is hoped that this procedure will facilitate the creation of computer aided design software for spatial $4C$ mechanisms.

8 Acknowledgments

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A The Distance from a Congruence Plane to a Sphere

Here we determine the minimum or normal distance d_{min} from a plane π of the congruence to the center of the bounding sphere $\mathbf{p}$. Moreover, we determine the point p_{min} in the plane nearest the sphere center $\mathbf{p}$.

First, let us define the direction and moment vectors of the lines that define π as,

$$L_1 = \begin{bmatrix} \mathbf{l}_1 \\ \mathbf{l}_1^0 \end{bmatrix} \tag{20}$$

and,

$$L_2 = \begin{bmatrix} \mathbf{l}_2 \\ \mathbf{l}_2^0 \end{bmatrix} \tag{21}$$

recall that $\mathbf{l}_1 = \mathbf{l}_2$. We chose to represent the plane π by the implicit equation,

$$(\mathbf{p}_\pi - \mathbf{p}_0) \cdot \mathbf{n} = 0 \tag{22}$$

where $\mathbf{n}$ is the normal vector to the plane, $\mathbf{p}_0$ is a given point in the plane, and $\mathbf{p}_\pi$ is a general point in the plane. From L_1 and L_2 we determine $\mathbf{n}$ and $\mathbf{p}_0$ as,

$$\mathbf{n} = \frac{(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{l}_1}{\|(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{l}_1\|} = \frac{(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{l}_2}{\|(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{l}_2\|} \tag{23}$$

$$\mathbf{p}_0 = \mathbf{p}_1 = \mathbf{l}_1 \times \mathbf{l}_1^0 \text{ or } \mathbf{p}_0 = \mathbf{p}_2 = \mathbf{l}_2 \times \mathbf{l}_2^0 \tag{24}$$

The normal distance d from the origin to the plane π is,

$$d = \mathbf{p}_0 \cdot \mathbf{n} \tag{25}$$

We can now express $\mathbf{p}_0$ in terms of d,

$$\mathbf{p}_0 = d\mathbf{n} \tag{26}$$

and substitute into Eq. 22 to yield,

$$\mathbf{p}_\pi \cdot \mathbf{n} = d \tag{27}$$

Eq. 27 yields an efficient method of determining if a general point $\mathbf{p}$ lies in the plane π. If $\mathbf{p} \cdot \mathbf{n} = d \pm \epsilon$, where epsilon is some small tolerance value, then $\mathbf{p}$ is said to lie in the plane π. If $\mathbf{p} \cdot \mathbf{n} > d + \epsilon$, then $\mathbf{p}$ is said to lie in the right half-space defined by π. Similarly, if $\mathbf{p} \cdot \mathbf{n} < d - \epsilon$, then $\mathbf{p}$ is said to lie in the left half-space. Moreover, the minimum distance d_{min} from the plane π of the congruence, defined by L_1 and L_2, to the center of the bounding sphere $\mathbf{p}$ is given by,

$$d_{min} = |\mathbf{p} \cdot \mathbf{n} - d| \tag{28}$$

Finally, we determine the point $\mathbf{p}_{min}$ in π which is closest to the center of the sphere $\mathbf{p}$ as,

$$\mathbf{p}_{min} = \mathbf{p} - (\mathbf{p} \cdot \mathbf{n} - d)\mathbf{n} \tag{29}$$