A Distance Metric for Finite Sets of Rigid-Body Displacements via the Polar Decomposition

Pierre M. Larochelle
Mechanical & Aerospace Engineering Department,
Florida Institute of Technology,
Melbourne, FL 32901-6975
e-mail: pierrel@fit.edu

Andrew P. Murray
Mechanical & Aerospace Engineering Department,
University of Dayton,
Dayton, OH 45469-0238

Jorge Angeles
Department of Mechanical Engineering,
McGill University,
Montreal, Quebec, H3A 2A7 Canada

An open research question is how to define a useful metric on the special Euclidean group SE(n) with respect to: (1) the choice of coordinate frames and (2) the units used to measure linear and angular distances that is useful for the synthesis and analysis of mechanical systems. We discuss a technique for approximating elements of SE(n) *with elements of the special orthogonal group* SO(n+1). *This technique is based on using the singular value decomposition (SVD) and the polar decompositions (PD) of the homogeneous transform representation of the elements of* SE(n). *The embedding of the elements of* SE(n) *into* SO(n+1) *yields hyperdimensional rotations that approximate the rigid-body displacements. The bi-invariant metric on* SO(n+1) *is then used to measure the distance between any two displacements. The result is a left invariant PD based metric on* SE(n).
[DOI: 10.1115/1.2735640]

1 Introduction

Simply stated, a metric measures the distance between two points in a set. There exist numerous useful metrics for defining the distance between two points in Euclidean space; however, defining similar metrics for determining the distance between two locations of a finite rigid body is still an area of ongoing research. In Ref. [1] Kazerounian and Rastegar define object norms that depend upon the volume or shape of the rigid-body being displaced. Geometrically motivated metrics that depend upon the displacement of the rigid body rather than its shape were proposed by Refs. [2,3]. Gupta investigates Euclidean error measures on rigid-body displacements [4]. In Ref. [5] a Lie group theory approach is taken and a left-invariant metric parameterized by a characteristic length is presented. Lin and Burdick present formal and practical conditions for kinematic metrics [6]. Moreover, in Ref. [7] addresses the inherent pitfalls when defining rigid-body displacement metrics. Fanghella and Galletti present metric relations based upon the closure equations associated with the kinematic chain producing the motion [8]. A local metric based on an optimized local mapping of the study of quadric via stereographic projection was proposed in Ref. [9]. Chirikjian presents convolution metrics on the group of rigid-body motions [10] and in Ref. [11] he compares and contrasts some metric methods. In Ref. [12] Belta and Kumar present a SVD based rigid-body motion interpolation.

In the cases of two locations of a finite rigid body in either SE(3) (spatial locations) or SE(2) (planar locations) any (Riemannian) metric used to measure the distance between the locations yields a result which depends upon the chosen reference frames (see Refs. [5,7]). However, for the specific case of orienting a finite rigid body in SO(n) bi-invariant metrics do exist. For example, Ravani and Roth [13] defined the distance between two orientations of a rigid body in space as the magnitude of the difference between the associated quaternions; a proof that this metric is bi-invariant may be found in Ref. [2]. One useful and easily evaluated metric d on SO(n) follows. Given two elements $[A_1]$ and $[A_2]$ of SO(n) we can define a metric using the Frobenius norm as

$$d = \|[I] - [A_2][A_1]^T\|_F \tag{1}$$

It is straightforward to verify that this is a valid metric on SO(n), see Ref. [14].

In Ref. [2] Larochelle and McCarthy proposed an algorithm for approximating displacements in SE(2) with orientations in SO(3). By building upon the work of Ravani and Roth [13], they arrived at a metric for planar locations in which the error induced by the spherical approximation is on the order of $1/R^2$, where R is the radius of the approximating sphere. Their algorithm is based upon an algebraic formulation which utilizes Taylor series expansions of sine() and cosine() terms in homogeneous transforms [15]. Etzel and McCarthy [16] later extended this work to spatial displacements by using orientations in SO(4) to approximate locations in SE(3).

This paper discusses an efficient alternative methodology for defining a metric on a finite set of elements in SE(n). Here, the underlying geometrical motivations are the same—to approximate displacements with hyperspherical rotations. However, we utilize the polar decomposition to yield hyperspherical orientations that approximate planar and spatial finite displacements. The work reported here is built upon ideas found in Refs. [17,18,15,19].

2 The SVD-Based Embedding

The SVD-based approach, analogous to the works summarized above, also uses hyperdimensional rotations to approximate displacements. However, this technique uses products derived from the SVD of the homogeneous transform to realize the embedding of SE(n−1) into SO(n) [20].

Consider the space of $n\times n$ matrices as shown in Fig. 1. Let $[T]$ be a $n\times n$ homogeneous transform that represents an element of SE(n−1). Note that $[T]$ defines a point in R^{n^2}. $[A]$ is the desired element of SO(n) nearest $[T]$ when it lies in a direction orthogonal to the tangent plane to SO(n) at $[A]$. The following theorem, based upon related works by Hanson and Norris [21], provides the foundation for the embedding,

Theorem 2.1. *Given any* $n\times n$ *matrix* $[T]$, *the closest element of SO(n) is given by*: $[A]=[U][V]^T$ *where* $[T]=[U]\times[diag(s_1,s_2,\ldots,s_n)][V]^T$ *is the SVD of* $[T]$.

Shoemake and Duff [19] prove that matrix $[A]$ satisfies the following optimization problem: *Minimize*: $\|[A]-[T]\|_F^2$ subject to: $[A]^T[A]-[I]=[0]$, where $\|[A]-[T]\|_F^2=\Sigma_{i,j}(a_{ij}-t_{ij})^2$ is used to denote the Frobenius norm. Since $[A]$ minimizes the Frobenius norm in R^{n^2}, it is the element of SO(n) that lies in a direction orthogonal to the tangent plane of SO(n) at $[R]$. Hence, $[A]$ is the closest element of SO(n) to $[T]$. Moreover, for full-rank matrices the SVD is well defined and unique. We now restate Theorem 2.1

Contributed by the Mechanisms and Robotics Committee of ASME for publication in the Journal of Mechanical Design. Manuscript received September 10, 2005; final manuscript received July 13, 2006. Review conducted by Kwun-Lon Ting. Paper presented at the ASME 2005 Design Engineering Technical Conferences and Computers and Information in Engineering Conference (DETC2005), Long Beach, CA, USA, September 24–28, 2005.

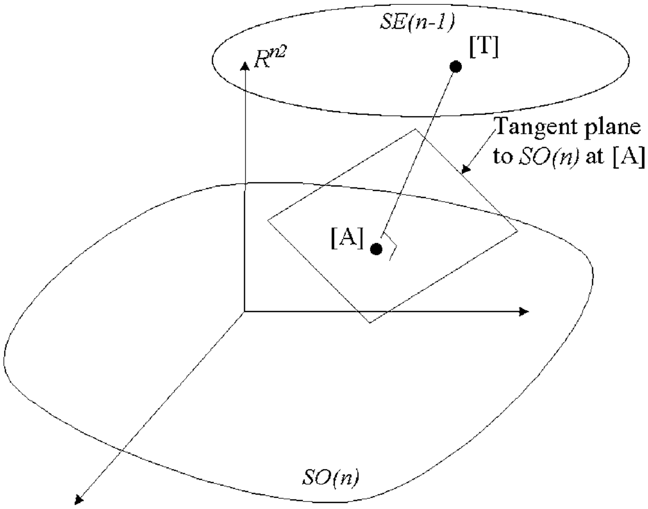

Fig. 1 The embedding of elements of SE($n-1$) in SO(n)

with respect to the desired SVD based embedding of SE($n-1$) into SO(n).

Theorem 2.2. *For* $[T]\in SE(n-1)$ *and* $[T]=[U]\times[diag(s_1,s_2,\ldots,s_n)][V]^T$, *if* $[A]=[U][V]^T$, *then* $[A]$ *is the unique element of* $SO(n)$ *nearest* $[T]$.

Recall that $[T]$, the homogenous representation of SE(n), is of full rank [22] and, therefore, $[A]$ exists, is well defined, and unique.

3 The PD-Based Embedding

The polar decomposition (PD) though perhaps less known than the SVD, is quite powerful and actually provides the foundation for the SVD [23]. Cauchy's polar decomposition theorem states that "a nonsingular matrix equals an orthogonal matrix either pre- or post-multiplied by a positive definite symmetric matrix" [24]. With respect to our application, for $[T]\in$ SE($n-1$) its PD is $[T]=[P][Q]$, where $[P]$ and $[Q]$ are $n\times n$ matrices such that $[P]$ is orthogonal and $[Q]$ is positive definite and symmetric. Recalling the properties of the SVD, the decomposition of $[T]$ yields $[U]\times[\mathrm{diag}(s_1,s_2,\ldots,s_{n-1})][V]^T$, where matrices $[U]$ and $[V]$ are orthogonal and matrix $[\mathrm{diag}(s_1,s_2,\ldots,s_{n-1})]$ is positive definite and symmetric. Moreover, it is known that for full rank square matrices the PD and the SVD are related by: $[P]=[U][V]^T$ and $[Q]=[V][\mathrm{diag}(s_1,s_2,\ldots,s_{n-1})][V]^T$ [23]. Hence, for $[A]=[U][V]^T$ we have $[A]=[P]$ and conclude that the polar decomposition yields the same element of SO(n). We now restate Theorem 2.2 with respect to the desired PD based embedding of SE($n-1$) onto SO(n).

Theorem 3.1. *If* $[T]\in SE(n-1)$ *and* $[P]$ *&* $[Q]$, *are the PD of* $[T]$ *such that* $[T]=[P][Q]$, *then* $[P]$ *is the unique element of SO(n) nearest* $[T]$.

Dubrulle [25] provides an algorithm for computing the PD that produces monotonic convergence in the Frobenius norm that "… generally delivers an IEEE double-precision solution in ~10 or fewer steps."

4 Implementation of the Metric

The PD-based embedding of SE($n-1$) into SO(n) reviewed above could be used for the systematic embedding of elements of SE($n-1$) into SO(n). However, to yield a useful metric for a finite set of displacements appropriate for design, the principal axes frame and the characteristic length are introduced.

4.1 The Principal Axes Frame. We now consider a finite set of n displacements ($n\geq 2$) and seek their magnitudes. In order to yield a left-invariant metric, we build upon the work of Kazerounian and Rastegar [1] in which approximately bi-invariant metrics were defined for a prescribed finite rigid body. Here, to avoid cumbersome volume integrals over the body we utilize a unit point mass model for the moving body, the rationale being that the moving frame in the application areas considered has some inherent importance. For example, in robot end-effector applications the moving frame will often be defined with its origin at the tool center point. Moreover, in motion synthesis tasks, the moving frame is often defined with its origin at the point on the moving body whose motion is critical to the task at hand.

We proceed by determining the position vector of the center of mass $\vec{c}$ and the principal axes frame PF associated with the n prescribed locations, where a unit point mass is located at the origin of each location

$$\vec{c}=\frac{1}{n}\sum_{i=1}^{n}\vec{d}_i \tag{2}$$

where $\vec{d}_i$ is the translation vector associated with the ith location (i.e., the origin of the ith location with respect to the fixed frame). Next, we define PF with its axes defined as the principal axes of the inertia tensor $[\mathbf{I}]$ of the n point mass system about the centroid $\vec{c}$. First, we determine the inertia tensor $[\mathbf{I}]$ associated with the n point mass system

$$[\mathbf{I}]=[1]\sum_{i=1}^{n}\|\vec{d}_i\|^2-\sum_{i=1}^{n}\vec{d}_i\vec{d}_i^T \tag{3}$$

where $[1]$ is the 3×3 (spatial) or 2×2 (planar) identity matrix.

Finally, we determine the principal axes frame PF

$$\mathrm{PF}=\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{c}\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{4}$$

where $\vec{v}_i$ are the unit eigenvectors of the inertia tensor $[\mathbf{I}]$. The directions of the vectors along the principal axes ($\vec{v}_i$) are chosen such that PF is a right-handed system. The center of mass and the principal axes frame are unique for the mechanical system and invariant with respect to both the choice of coordinate frames and the system of units [27,28]. Note that the principal frame is not dependent on the orientations of the frames at hand—only the positions of their origins. However, the metric is dependent on the orientations of the frames.

4.2 Characteristic Length. In order to resolve the unit disparity between translations and rotations we use a characteristic length to normalize the translational terms in the displacements. There are three general approaches to selecting a characteristic length: based upon the body being displaced [1,7], based upon the kinematic chain generating the motion [8], or based upon the motion task [29,5]. The characteristic length we chose is $R=24L/\pi$, where L is the maximum translational component in the set of displacements at hand. This formulation is based upon the motion task and is reported in Refs. [2,16]. This characteristic length is the radius of the hypersphere that approximates the translational terms by angular displacements that are ≤ 7.5 deg. It was shown in Ref. [30] that this radius yields an effective balance. Note that the metric presented here is not dependent upon this particular choice of characteristic length and that, if so desired, an alternative formulation may be utilized.

4.3 Step by Step. We now summarize the implementation of the methodology. For a set of n locations, proceed as follows:

1. Determine PF associated with the n displacements;
2. Determine the relative displacements from PF to each of the n locations;

Table 1 Eleven planar locations

No.	a	b	α (deg)	$\lVert[T]\rVert$
1	−1.0000	−1.0000	90.0000	0.1237
2	−1.2390	−0.5529	77.3621	0.3214
3	−1.4204	0.3232	55.0347	0.8455
4	−1.1668	1.2858	30.1974	1.4039
5	−0.5657	1.8871	10.0210	1.8118
6	−0.0292	1.9547	1.7120	1.9641
7	0.2632	1.5598	10.0300	1.8109
8	0.5679	0.9339	30.1974	1.4023
9	1.0621	0.3645	55.0346	0.8434
10	1.6311	0.0632	77.3620	0.3214
11	2.0000	0.0000	90.0000	0.1350

3. Determine the characteristic length R associated with the n relative displacements and scale the translation terms in each by $1/R$;
4. Compute the elements of SO(3)(planar) or SO(4)(spatial) associated with PF and each of the scaled relative displacements using Theorem 3.1; and
5. The magnitude of the ith displacement is defined as the distance from PF to the ith scaled relative displacement as computed via Eq. (1). The distance between any two of the n locations is similarly computed via the application of Eq. (1) to the scaled relative displacements embedded in SO(3) or SO(4).

We note that since the center of mass and PF are invariant with respect to both the choice of coordinate frames and the system of units [27,28], that the polar decomposition displacement metric is left invariant. The subsequent examples illustrate the application of the above methodology to a finite set of planar or spatial displacements.

5 Example 1

Consider the 11 planar locations that define a motion generation task proposed by J. Michael McCarthy of U.C. Irvine for the 2002 ASME Mechanisms & Robotics Conference and found in Ref. [26]. The 11 (a,b,α) locations are listed in Table 1 and shown in Fig. 2 along with the fixed reference frame F where the x axes are shown in red(dark) and the y axes in green(light). We proceed as above and determine PF

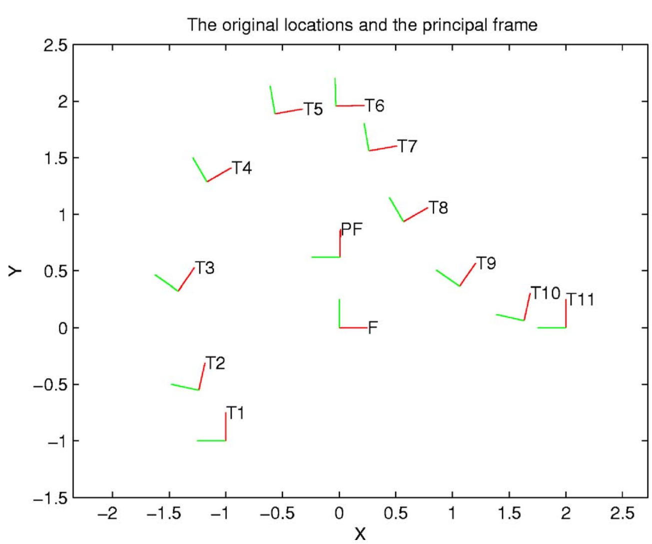

Fig. 2 The 11 planar locations and PF

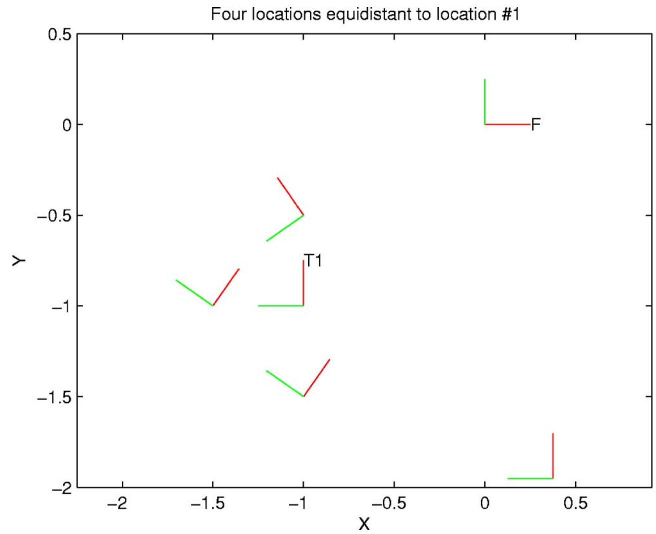

Fig. 3 The fixed frame and four locations equidistant to location No. 1

$$\mathrm{PF}=\begin{bmatrix} 0.0067 & -1.0000 & 0.0094 \\ 1.0000 & 0.0067 & 0.6199 \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

The 11 locations are now determined with respect to PF and the maximum translational component is found to be 1.4278 and the associated characteristic length is $R=24\times 1.4278/\pi=10.9073$. Finally, the magnitude of each of the displacements is computed via Eq. (1) and listed in Table 1. To illustrate the applicabilility of the metric to tasks such as motion, synthesis, motion, interpolation, etc., we show four arbitrary locations that are equidistant to location No. 1 in Fig. 3.

6 Example 2

Consider four spatial locations from the rigid-body motion generation example presented in Ref. [31]. The four locations are listed in Table 2, and their associated PF is

$$\mathrm{PF}=\begin{bmatrix} -0.5692 & 0.8061 & -0.1617 & 0.75000 \\ -0.7807 & -0.5916 & -0.2012 & 1.5000 \\ -0.2578 & 0.0117 & 0.9661 & 0.4375 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

Next, the four locations with respect to the principal frame are determined. The maximum translational component is found to be 1.7108 and the associated characteristic length is $R=13.0695$. Finally, the magnitude of each of the displacements is listed in Table 2. Note that the magnitude of the first location is not zero because the relative displacement from PF to the first location is nonidentity.

Table 2 Four spatial locations

No.	x	y	z	θ (deg)	ϕ (deg)	ψ (deg)	$\lVert[T]\rVert$
1	0.00	0.00	0.00	0.0	0.0	0.0	2.5281
2	0.00	1.00	0.25	15.0	15.0	0.0	2.5701
3	1.00	2.00	0.50	45.0	60.0	0.0	2.7953
4	2.00	3.00	1.00	45.0	80.0	0.0	2.8057

7 Conclusions

We discussed a methodology for measuring distances on a finite set of elements of SE(n). This technique is based on embedding SE(n) into SO($n+1$) via either the polar or the singular value decompositions of the homogeneous transform representation of SE(n). A bi-invariant metric on SO($n+1$) is then used to measure the *distance* between any two displacements SE(n). The resulting distance measure was shown to be left invariant. A detailed methodology for applying this technique was presented and illustrated by two examples.

ACKNOWLEDGMENT

This material is based upon work supported by the National Science Foundation under Grants Nos. 0422705 and 0422731. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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