

# A Coordinate Frame Useful for Rigid-Body Displacement Metrics

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*This paper presents the definition of a coordinate frame, entitled the principal frame (PF), that is useful for metric calculations on spatial and planar rigid-body displacements. Given a set of displacements and using a point mass model for the moving rigid-body, the PF is determined from the associated centroid and principal axes. It is shown that the PF is invariant with respect to the choice of fixed coordinate frame as well as the system of units used. Hence, the PF is useful for left invariant metric computations. Three examples are presented to demonstrate the utility of the PF. [DOI: 10.1115/1.4002245]*

## 1 Introduction

The focus here is on presenting a methodology for identifying a useful fixed frame for performing metric computations on finite sets of planar or spatial displacements. A metric is used to measure the distance between two points in a set. There are various metrics for finding the distance between two points in Euclidean space. However, finding the distance between two locations of a rigid-body is still the subject of ongoing research, see Refs. [1–13]. Kazerounian and Rastegar [13] defined metrics that depend on the shape and mass density of the finite moving body. More recently, Sharf et al. [14] discussed Riemannian and Euclidean averages for rigid-body rotations and Angeles [15] investigated the use of characteristic lengths that are used to combine translations and rotations in some manner for use in distance metrics. Furthermore, Zhang and Ting [16] examined Riemannian metrics on point-line displacements. Finally, Di Gregorio [17] sought to employ a geometric approach to identify useful distance metrics.

For two locations of a rigid-body (either SE(2)-planar or SE(3)-spatial) all metrics yield a distance, which is dependent on the chosen fixed or moving frames of reference and the units used, see Refs. [4,7]. But, a metric independent of these choices, referred to as bi-invariant, is desirable. Metrics independent of the choice of coordinate frames and the units used do exist on SO(N), see the work of Larochelle and McCarthy [8]. One bi-invariant metric defined by Ravani and Roth [18] defines the distance between two orientations of a rigid-body as the magnitude of the difference between the associated quaternions. Of related background interest are the works by Horn [19] and Shoemake and Duff [20]. Horn solved the problem of finding the rigid-body transformation between two coordinate frames using point coordinates by using Hamilton's quaternions, whereas Shoemake and Duff examined the problem of decomposing homogeneous rigid-body displacements into rotations and translations using the polar decomposition.

The PF has been introduced to support the ongoing development of polar decomposition based metrics on the displacement groups [21–23]. These techniques are based on the polar decomposition (PD) of the homogenous transform representation of the elements of SE(N) and the principal frame (PF) associated with the finite set of rigid-body displacements. The mapping of the elements of the special Euclidean group SE(N-1) to SO(N) yields hyperdimensional rotations that approximate the rigid-body displacements. Conceptual representations of the mapping of SE(N-1) to SO(N) are shown in Figs. 1 and 2. Once the elements are mapped to SO(N) distances can then be evaluated by using a bi-invariant metric on SO(N). Due to the use of the PF, the resulting metric on SE(N-1) is left invariant (i.e., independent of the choice of fixed frame F).

## 2 Metric on SO(N)

Here, we briefly review the use of the polar decomposition to yield the hyperdimensional rotations that approximate spatial or planar displacements. The elements in SO(N) are derived from homogenous transformations representing planar SE(2) or spatial SE(3) displacements by polar decompositions, as shown in Fig. 2 and derived in Refs. [21,24]. The distance between any two elements  $[A_1]$  and  $[A_2]$  in SO(N) is determined by using the Frobenius norm as follows:

$$d = \|[I] - [A_2][A_1]^T\|_F \quad (1)$$

It has been verified that this is a valid metric on SO(N), see Ref. [25].

## 3 Finite Sets of Locations

As was reviewed in Sec. 2, the elements of SE(N-1) can be approximated by elements of SO(N) by using the polar decomposition. However, the metric on SO(N) will not be well defined because of its dependence on the choice of fixed reference frame. In order to yield a useful metric for a finite set of displacements, the principal frame (PF) is introduced. The principal frame is unique for a finite set of displacements and invariant with respect to the choice of fixed coordinate frame and the system of units, see Refs. [26,27]. All of the displacements are then expressed with respect to the principal frame and all distances are measured with respect to this same frame. Hence, the polar decomposition based metric yields results that are invariant with respect to the choice of fixed frame. Next, we present the detailed implementation of this methodology.

Consider the case when a finite number of  $n$  displacements ( $n \geq 2$ ) are given and we have to find the magnitude of these displacements. The displacements depend on the coordinate frames and the system of units chosen. In order to yield a left invariant metric, we utilize a PF that is derived from a unit point mass model for a moving body as suggested by Larochelle [24]. A unit point mass is assigned to the origin of each of the coordinate frames representing rigid-body displacements, as shown in Fig. 3. The point masses are then used to determine the center of mass of the system and, eventually, the invariant principal frame of the set of displacements, as shown in Fig. 4. This is done to yield a metric that is independent of the geometry and mass distribution of the moving body. The center of mass and the principal frame are unique for the system and invariant with respect to both the choice of fixed coordinate frame and the system of units [26,27].

The procedure for determining the center of mass  $c$  and the PF associated with the  $n$  prescribed locations is now described. A unit point mass is located at the origin of each location, as shown in Figs. 3 and 4.

Contributed by the Mechanisms and Robotics Committee of ASME for publication in the JOURNAL OF MECHANISMS AND ROBOTICS. Manuscript received July 25, 2008; final manuscript received July 19, 2010; published online September 30, 2010. Assoc. Editor: Frank C. Park.

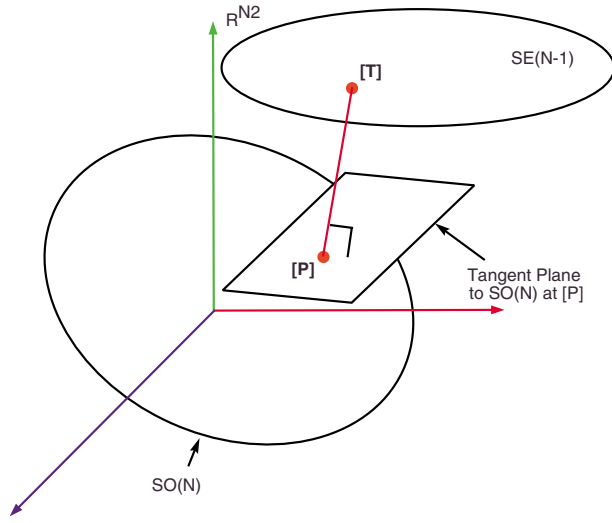


Fig. 1 SE(N-1) to SO(N)

$$c = \frac{1}{n} \sum_{i=1}^n d_i \quad (2)$$

where  $d_i$  is the translation vector associated with the  $i$ th location (i.e. the origin of the  $i$ th location with respect to F).

The  $PF$  is defined such that its axes are aligned with the principal axes of the  $n$  point mass system and its origin is at the centroid  $c$ . After finding the centroid of the system, we determine the principal axes of the point mass system as follows. The inertia tensor is computed from

$$[I] = [1] \sum_{i=1}^n \|d_i\|^2 - \sum_{i=1}^n d_i d_i^T \quad (3)$$

where  $[1]$  is the  $3 \times 3$  (spatial) or  $2 \times 2$  (planar) identity matrix. The principal frame is then determined to be

$$[PF] = \begin{bmatrix} v_1 & v_2 & v_3 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

where  $v_i$  are the principal axes (eigenvectors) associated with the inertia tensor  $[I]$ , see the work of Greenwood [26]. The directions of the vectors along the principal axes  $v_i$  are chosen such that the principal frame is a right-handed system. However, Eq. (4) does not uniquely define the  $PF$  since the eigenvectors  $v_i$  of the inertia

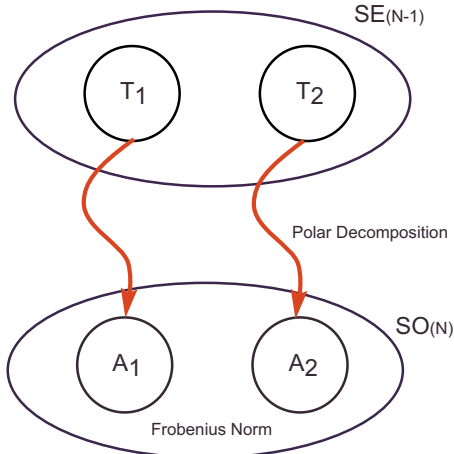


Fig. 2 Mapping to SO(N)

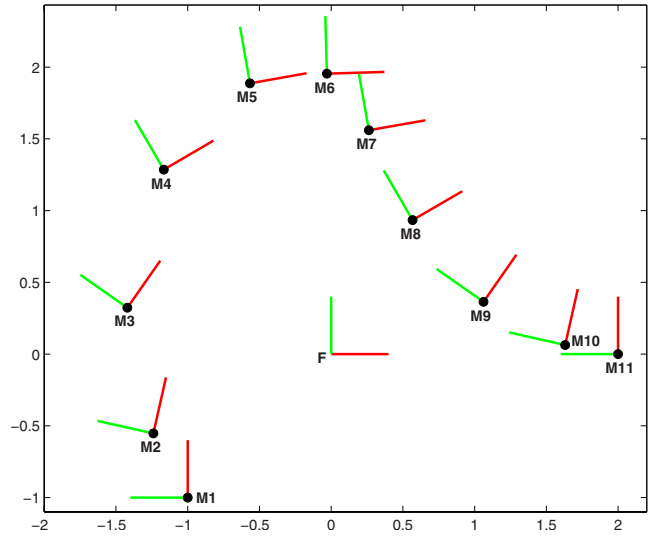


Fig. 3 Unit point mass model

tensor are not unique; i.e., both  $v_i$  and  $-v_i$  are eigenvectors associated with  $[I]$ . In order to resolve this ambiguity and yield a unique  $PF$ , we choose the eigenvector directions that most closely aligned  $PF$  to F. Note that in the planar case, the  $PF$  reduces to a  $3 \times 3$  matrix

$$[PF] = \begin{bmatrix} v_1 & v_2 & c \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

The eight different right-handed  $PF$  orientations that are possible in the spatial case are

$$[v_1 \ v_2 \ v_3]$$

$$[v_2 \ -v_1 \ v_3]$$

$$[-v_1 \ -v_2 \ v_3]$$

$$[-v_2 \ v_1 \ v_3]$$

$$[v_2 \ v_1 \ -v_3]$$

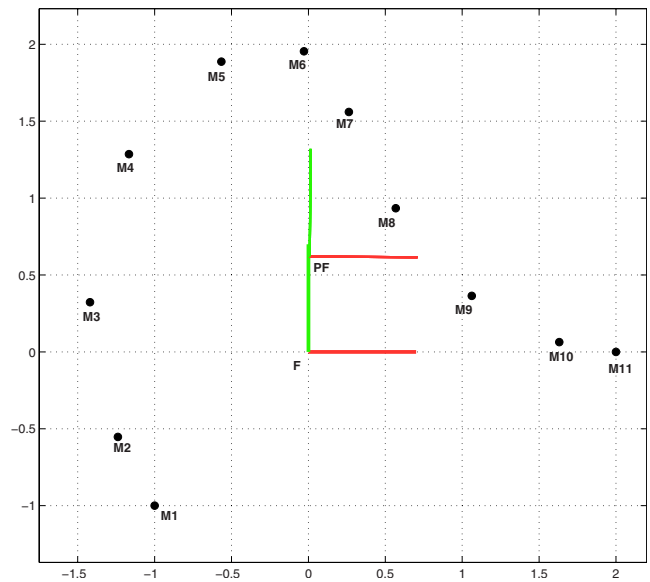


Fig. 4 Unit point mass model and associated  $PF$

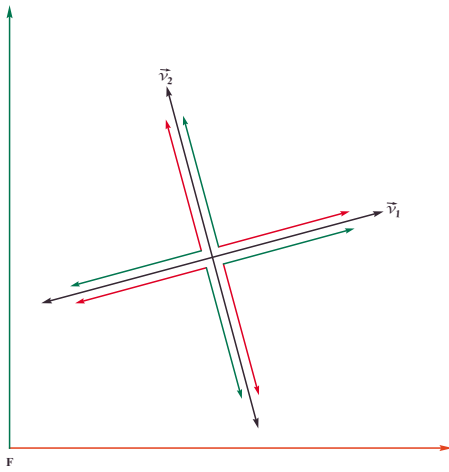


Fig. 5 Four possible orientations for the *PF*

$$[\mathbf{v}_1 \ -\mathbf{v}_2 \ -\mathbf{v}_3]$$

$$[-\mathbf{v}_2 \ -\mathbf{v}_1 \ -\mathbf{v}_3]$$

$$[-\mathbf{v}_1 \ \mathbf{v}_2 \ -\mathbf{v}_3]$$

and in the planar case there are four possible orientations of the *PF*, as seen in Fig. 5,

$$[\mathbf{v}_1 \ \mathbf{v}_2]$$

$$[\mathbf{v}_2 \ -\mathbf{v}_1]$$

$$[\mathbf{v}_1 \ -\mathbf{v}_2]$$

$$[-\mathbf{v}_2 \ \mathbf{v}_1]$$

The *PF* is selected as the frame that is most closely oriented, per Eq. (1), to the fixed frame. However, there are degenerate cases that must now be addressed.

In the degenerate planar case in which the lines defining the *PF* form equal angles (i.e., 45 deg) with the axes of the fixed frame, then all four possible orientations will be equidistant to the fixed frame's orientation. In this case, we define the *x*-axis of the *PF* along the eigenvector in the first quadrant. In the similarly degenerate spatial case, in which the eigenvectors form equal angles (i.e., 54.7 deg) with each axis of the fixed frame, we define the *x*-axis of the *PF* along the eigenvector in the first octant and the *y* and *z* axes are chosen such that *PF* is the frame that is most closely oriented, per Eq. (1), to the fixed frame.

#### 4 Summary of the *PF* and *PD* Metric Technique

For a set of *n* finite rigid-body locations the steps to be followed are as follows:

1. Determine the *PF* associated with the *n* locations.
2. Determine the relative displacements from the *PF* to each of the *n* locations.
3. Determine the characteristic length *R* associated with the *n* displacements with respect to the *PF* and scale the translation terms in each by 1/*R*.
4. Compute the projections of *PF* and each of the scaled relative displacements using the polar decomposition.
5. The magnitude of the displacement is defined as the distance from the *PF* to the scaled relative displacement as computed via Eq. (1). The distance between any two of the *n* locations is similarly computed by the application of Eq. (1) to the projected scaled relative displacements.

Table 1 11 planar locations

<i>x</i>	<i>y</i>	$\psi$ (deg)	Mag.
-1.0000	-1.0000	90.0000	2.0076
-1.2390	-0.5529	77.3621	1.7762
-1.4204	0.3232	55.0347	1.3165
-1.1668	1.2858	30.1974	0.7483
-0.5657	1.8871	10.0210	0.2644
-0.0292	1.9547	1.7120	0.0807
0.2632	1.5598	10.0300	0.2606
0.5679	0.9339	30.1974	0.7464
1.0621	0.3645	55.0346	1.3159
1.6311	0.0632	77.3620	1.7762
2.0000	0.0000	90.0000	2.0078

#### 5 Example: 11 Planar Locations

Consider the rigid-body guidance problem proposed by J. Michael McCarthy, U.C. Irvine for the 2002 ASME International Design Engineering Technical Conferences held in Montreal, Quebec, as shown in Fig. 3 [28]. The 11 planar locations are listed in Table 1 and the origins of the coordinate frames with respect to the fixed reference frame *F* are shown in Fig. 4. The centroid and the principal axes directions are calculated and used to determine the *PF* (Fig. 6)

$$[PF] = \begin{bmatrix} 1.0000 & 0.0067 & 0.0094 \\ -0.0067 & 1.0000 & 0.6199 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix} \quad (6)$$

Note that this *PF* differs from that originally reported in Ref. [21] due to the more rigorous methodology presented here. The 11 locations are now determined with respect to the *PF* and the maximum translational component is found to be 1.9947 and the resulting characteristic length is  $R=24L/\pi=15.239$ .

To illustrate the utility of the *PF*, we present its application for distance metric calculations per Ref. [21]. The 11 locations are then scaled by the characteristic length in order to find the distance to the principal frame. The magnitude of each of the displacements with respect to the *PF* is listed in Table 1. The distance between any two of the locations is computed by the

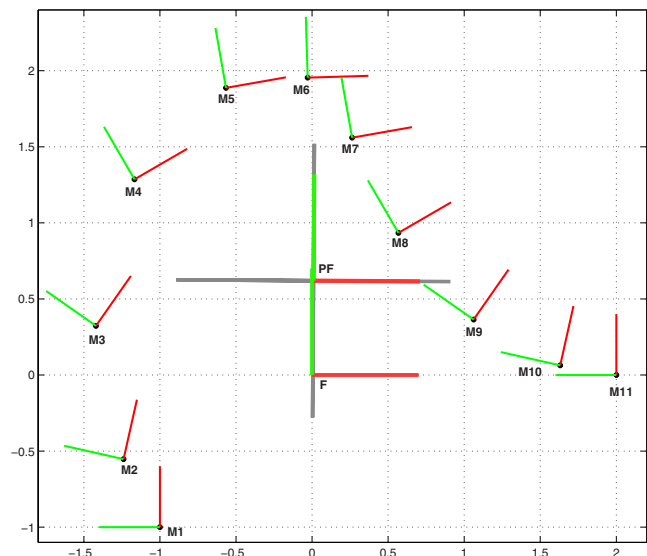


Fig. 6 Principal frame for eleven desired locations

**Table 2 Ten desired locations**

$x$	$y$	$z$	Long ( $\theta$ )	Lat ( $\phi$ )	Roll ( $\psi$ )
1	0	5	100	0	0
2	0	4	90	0	10
3	0	3	80	0	20
4	0	2	70	0	30
5	0	1	60	0	40
6	0	-1	50	0	50
7	0	-2	40	0	60
8	0	-3	30	0	70
9	0	-4	20	0	80
10	0	-5	10	0	90

application of Eq. (1) to the projected scaled relative displacements. For example, the distance between locations 1 and 2 was found to be 0.3115.

**6 Example: Ten Spatial Locations**

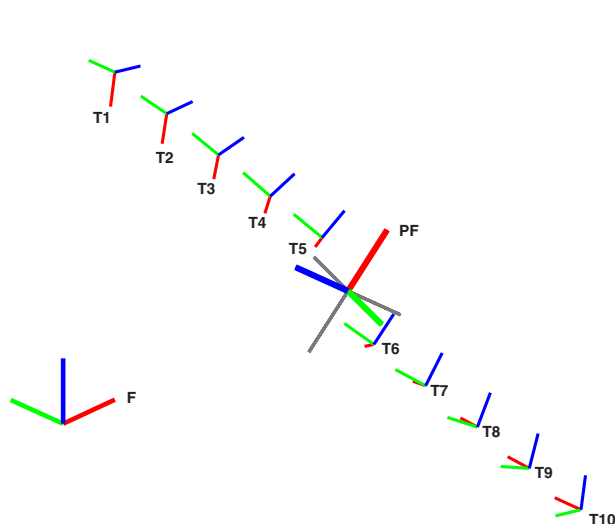
Consider the rigid-body guidance problem investigated by Larochelle [24]. The ten spatial locations with respect to the fixed reference frame  $F$  are listed in Table 2 and shown in Fig. 7. The principal frame is given by

$$[PF] = \begin{bmatrix} 0.756 & 0.655 & 0.000 & 5.500 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ 0.655 & -0.756 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 \end{bmatrix} \quad (7)$$

The maximum translational component  $L$  is found to be 6.7256 and the associated characteristic length is  $R=24L/\pi=51.3795$ . The distance from the first location to the principal frame was found to be 2.7488. The distance between locations 1 and 2 was found to be 0.3485.

**7 Example: Pick and Place Task**

Consider the following rigid-body guidance problem representing a spatial pick and place operation commonly found in an industrial assembly line. The ten spatial locations with respect to the fixed reference frame  $F$  are listed in Table 3 and shown in Fig. 8. The principal frame is given by



**Fig. 7 Principal frame for ten desired locations**

**Table 3 Ten desired locations**

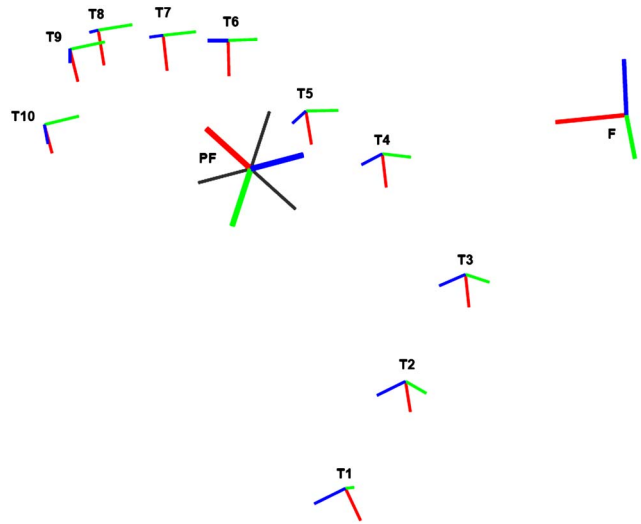
$x$	$y$	$z$	Long ( $\theta$ )	Lat ( $\phi$ )	Roll ( $\psi$ )
5.0	9.0	1.0	100	20	35
4.0	8.0	2.0	90	30	10
3.0	7.0	3.0	80	35	20
4.0	6.0	4.5	70	45	30
5.0	5.5	5.0	60	57	40
6.0	5.0	6.0	50	35	50
7.0	6.0	7.0	40	45	60
8.0	7.0	8.0	30	50	70
8.5	8.0	8.5	20	72	80
9.0	9.0	8.0	10	85	90

$$[PF] = \begin{bmatrix} 0.603 & 0.378 & -0.702 & 5.950 \\ 0.002 & 0.880 & 0.475 & 7.050 \\ 0.797 & -0.289 & 0.530 & 5.300 \\ 0.000 & 0.000 & 0.000 & 1.000 \end{bmatrix} \quad (8)$$

The maximum translational component  $L$  is found to be 4.0920 and the associated characteristic length is  $R=24L/\pi=31.2602$ . The distance from the first location to the principal frame was found to be 2.8135. The distance between locations 1 and 2 was found to be 0.7842.

**8 Conclusions**

A coordinate frame, entitled the principal frame ( $PF$ ), that is useful for metric calculations on spatial and planar rigid-body displacements was rigorously defined and presented. Given a set of displacements and using a point mass model for the moving rigid-body, the  $PF$  is determined from the associated centroid and principal axes. It was shown that the  $PF$  is left invariant, i.e., independent of both the choice of the fixed coordinate frame and the system of units used. Hence, by defining relative displacements with respect to the  $PF$ , the  $PF$  has proven useful in obtaining left invariant distance metric computations. Three examples that utilized the  $PF$  and the polar decomposition based metric of Larochelle et al. [21] were presented to demonstrate the utility of the  $PF$ . The principal frame has potential applications in mechanism approximate motion synthesis, robot motion planning, and other applications, which benefit from left invariant metrics on the spatial and planar displacement groups.



**Fig. 8 Principal frame for the pick and place task**

## Acknowledgment

This material is based on the work supported by the National Science Foundation under Grant No. 0422705.

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