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USING OPTIMIZATION FOR THE MIXED EXACT-APPROXIMATE SYNTHESIS OF PLANAR MECHANISMS

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ABSTRACT

Exact synthesis algorithms for planar mechanisms for rigid-body guidance are limited by the number of poses the mechanism can position the rigid-body in Euclidean space. The mixed exact-approximate synthesis algorithm described guides a rigid exactly through two positions and approximately through n guiding positions. It breaks down a rigid-body guidance task into n subproblems of three positions to be solved by an exact synthesis algorithm. A novel algorithm utilizing MATLAB's constrained non-linear optimization tools is proposed. The algorithm can be utilized to find within a bounded design parameter space the RR dyad that exactly reaches two positions and minimizes the distance to n positions. Two such dyads can be synthesized independently and then combined to yield a planar four-bar mechanism. An example using the proposed algorithm to design a planar four-bar mechanism to solve McCarthy's 11-position synthesis problem stated at the 2002 ASME Conference is included.

INTRODUCTION

A mechanism is a mechanical device that has the purpose of transferring motion and/or force from a source to an output [1]. An RR dyad comprises of a rigid link and two revolute joints, as shown in Fig. 1 where *AB* represents the rigid link and *A* and *B* are the R joints. The fixed pivot **u**, is considered stationary with respect to a fixed reference frame [*F*]. The moving pivot **v**, is

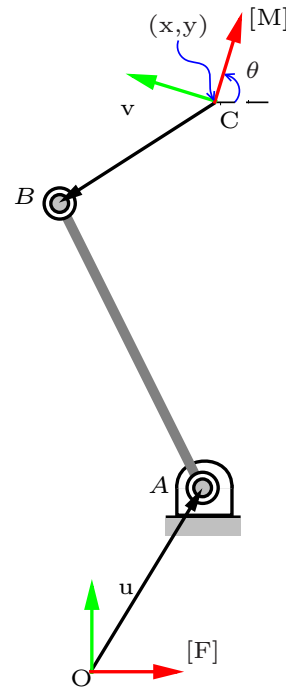**FIGURE 1**. PLANAR RR DYAD GEOMETRY AND NOMENCLATURE

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stationary with respect to a moving reference frame $[M]$. This RR dyad can be part of a planar linkage and used to perform various kinematic synthesis tasks such as function generation, path generation and rigid-body guidance. Considering the rigid-body guidance (RBG) problem, the RR dyad is required to guide a rigid-body through a sequence of precision positions or poses. Each position is a location in terms of Euclidean coordinates (x, y) and orientation (θ) with respect to the fixed reference frame $[F]$. An RR dyad can be synthesized using exact synthesis or mixed exact-approximate synthesis. Using classical Burmester Theory [1, 2] for exact synthesis, it is possible to determine an RR dyad that will guide a rigid-body exactly through a maximum of 5 prescribed poses. However, for more number of prescribed poses (>5), an exact synthesis technique cannot generate solutions and a mixed exact-approximate synthesis technique is utilized. In such scenarios, the RR dyad need only satisfy the exact positions and approximately guide the rigid-body for the remaining prescribed positions.

Related works

A modest amount of research has been done over the years towards developing mixed exact-approximate synthesis algorithms. One of the earliest works in mixed exact-approximation was undertaken by Mirth [3], who used the properties of Burmester curves to generate a solution space for three precision and n number of approximate positions. Mirth treated the problem as multiple four-location problems to yield multiple Burmester curves. The final solution space was obtained by intersecting the dyad solution space obtained from each four-location sub-problem. Holte et al. [4] developed mathematical techniques based on fuzzy constraints to generate overlapping solution spaces similar to [3]. Their work claimed that optimization was not required to find the solution space; rather it could be used to improve the designer's ability to find good working solutions. Mlinar and Erdman [5] introduced the notion of Burmester field generated by varying a design parameter to fit approximate positions. The workable linkage would be determined from the Burmester curves contained in the Burmester field.

Researchers have also explored optimization techniques using mixed exact-approximate synthesis to generate optimal linkage for motion generation. Bulatovic and Djordjevic [6] proposed a method of controlled deviations to control the motion of the coupler in a four-bar linkage. The Hooke-Jeeves' optimization method they use is a method of direct searching and does not use deductions of the objective function. Rather, it compares its values in each iteration and changes mechanism parameters in the direction of minimizing the value of the objective function. Smaili and Diab [7] discuss a closed path mixed-exact approximate synthesis algorithm based on optimum synthesis as opposed to using loop-closure equations. They utilized an objective function based on $\log_{10}$ of the error between the desired positions

and those generated by the optimum solution. Akhras and Angeles [8] discuss unconstrained non-linear least-squared techniques in the optimization of planar linkages for rigid-body guidance. They utilize a variable-separation technique to isolate configuration parameters and evaluate these parameters by formulating an unconstrained overdetermined system of nonlinear algebraic equations. Gogate and Matekar [9] use evolutionary methods with alternate error functions based on displacement of pole positions and fundamental geometrical properties of the four-bar mechanism for rigid-body guidance.

Organization of the Paper

This paper describes methods to synthesize optimal RR dyads that can be used to assemble a planar linkage for rigid-body guidance. The task is such that the linkage is required to guide the rigid-body exactly at the pick and place positions and approximately through n guiding positions. The RBG task is broken into n sub-problems of an exact synthesis algorithm that will exactly guide the rigid-body through three positions. The optimization process entails minimizing a multivariate objective function by starting at multiple initial points over a given set of design parameter space constraints. The objective function is a function of the outcome of the n sub-problems. MATLAB's [10] constrained non-linear optimization tool *fmincon* is used to synthesize optimal RR dyads. Two optimal RR dyads may be combined to form a optimal four-bar mechanism. Finally, the algorithm is tested against McCarthy's 11 position synthesis problem stated at the 2002 ASME Conference [11].

MIXED EXACT-APPROXIMATE SYNTHESIS

A mixed exact-approximate synthesis algorithm of an RR dyad for pick-and-place tasks is described in [12]. Consider the pick-and-place task of guiding a rigid-body through an exact pick and place position and approximately through n guiding positions. These positions may be denoted as $[M_i] = [A(\theta_i), \mathbf{d}_i]$, $i = pick, 1, \ldots, n, place$ with respect to a fixed reference frame $[F]$, where $A(\theta_i) \in \mathrm{SO}(2)$. The mixed exact-approximate synthesis of an RR dyad is broken into n sub-problems of a three position exact synthesis algorithm. Each sub-problem is solved as an exact synthesis task with the three positions $\{pick, i, place\}$ where, $i = 1, \ldots, n$.

Notation of an RR dyad

An RR dyad is characterized by a rigid link and two revolute pivots as shown in Fig. 1. For the exact synthesis of an RR dyad for three or more positions, the fixed pivot $\mathbf{u}$ and moving pivot $\mathbf{v}$ have a one-to-one onto mapping f with respect to each other i.e., $f : \mathbf{u} \leftrightarrow \mathbf{v}$. For the sake of generality, the notation $\mathbf{p}$ can represent either a fixed pivot or moving pivot i.e., say $\mathbf{p}$ is a single element of the set $P \in \{\mathbf{u}, \mathbf{v}\}$. The complimentary pivot of $\mathbf{p}$ is
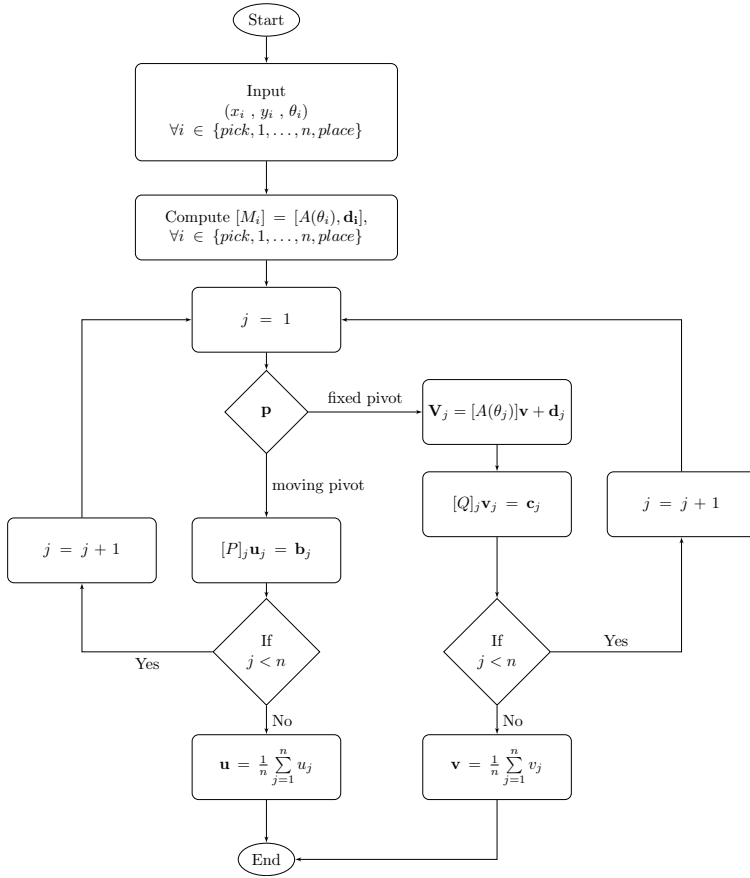
Start

Input
(x_i, y_i, θ_i)
$\forall i \in \{pick, 1, \ldots, n, place\}$

Compute $[M_i] = [A(\theta_i), \mathbf{d_i}]$,
$\forall i \in \{pick, 1, \ldots, n, place\}$

$j = 1$

p   fixed pivot $\mathbf{V}_j = [A(\theta_j)]\mathbf{v} + \mathbf{d}_j$

moving pivot

$[Q]_j\mathbf{v}_j = \mathbf{c}_j$ $j = j + 1$

$j = j + 1$ $[P]_j\mathbf{u}_j = \mathbf{b}_j$

If $j < n$ If $j < n$ Yes

Yes   No   No

$\mathbf{u} = \frac{1}{n}\sum_{j=1}^{n} u_j$ $\mathbf{v} = \frac{1}{n}\sum_{j=1}^{n} v_j$

End

FIGURE 2. ALGORITHM OF MIXED EXACT-APPROXIMATE SYNTHESIZE OF RR DYADS

$\mathbf{cp} = P \setminus \mathbf{p}$. An RR dyad can be denoted as $(\mathbf{p}, \mathbf{cp})$. The designer can choose $\mathbf{p}$ and determine the $\mathbf{cp}$ from a three position exact synthesis algorithm. Given $\mathbf{p}$, the n sub-problems of the mixed exact-approximate synthesis can be solved to obtain a set of $\mathbf{cp}_i$ which is later used in the objective function of the optimization process.

Exact Synthesis Algorithm

Planar four-bar mechanisms may be formed by connecting the end-link of two RR dyads [2]. Each such RR dyad is to be synthesized such that it reaches a set of three task positions given by the displacement $[M_i] = [A(\theta_i), \mathbf{d}_i]$, $i = 1, 2, 3$ with respect to a fixed reference frame $[F]$, where $A(\theta) \in SO(2)$. The moving pivot $\mathbf{v}$ with respect to $[M_i]$ can be represented as $\mathbf{V}_i$ in $[F]$ by $\mathbf{V}_i = [A(\theta_i)]\mathbf{v} + \mathbf{d}_i$. All such moving pivots $\mathbf{V}_i$ are constrained to

lie on a circle of radius d around the fixed pivot $\mathbf{u}$, that is,

$$(\mathbf{u} - \mathbf{V}_i) \cdot (\mathbf{u} - \mathbf{V}_i) = (\mathbf{u} - [A(\theta_i)]\mathbf{v} - \mathbf{d}) \cdot (\mathbf{u} - [A(\theta_i)]\mathbf{v} - \mathbf{d}) = d^2 \tag{1}$$

These equations yield a unique solution for either the fixed pivot $\mathbf{u}$ or the moving pivot $\mathbf{v}$ for an arbitrary choice of the other [2].

Select the moving pivot

Given a moving pivot $\mathbf{v}$ for three prescribed positions, there are many methods to solve for a fixed pivot $\mathbf{u}$. However, this paper focuses on Larochelle's method [12]. The constraint equation given in Eqn. (1) is applied to each task position to obtain three such equations. The first equation is subtracted from the remaining two to arrive at a linear system of equations:

$$[P]\mathbf{u} = \mathbf{b} \tag{2}$$

where,

$$[P] = \begin{bmatrix} 2(\mathbf{V}_2^T - \mathbf{V}_1^T) \\ 2(\mathbf{V}_3^T - \mathbf{V}_1^T) \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{V}_2^T\mathbf{V}_2 - \mathbf{V}_1^T\mathbf{V}_1^T \\ \mathbf{V}_3^T\mathbf{V}_3 - \mathbf{V}_1^T\mathbf{V}_1^T \end{bmatrix}.$$

The fixed pivot $\mathbf{u}$ of the dyad that reaches all three task positions can be computed by solving the above linear system using standard linear algebra techniques.

Select the fixed pivot

Given the fixed pivot $\mathbf{u}$ for three prescribed positions, the paper proposes the following method to solve for a moving pivot $\mathbf{v}$. The constraint Eqn. (1) is re-arranged to solve for the moving pivot $\mathbf{v}$ and applied to each task position $[M_i] = [A(\theta_i), \mathbf{d}_i]$. The first equation is subtracted from the remaining three constraint equations and the following linear system is obtained:

$$[Q]\mathbf{v} = \mathbf{c} \tag{3}$$

where,

$$[Q] = 2\begin{bmatrix} (\mathbf{u} - \mathbf{d}_2)^T[A_2] - (\mathbf{u} - \mathbf{d}_1)^T[A_1] \\ (\mathbf{u} - \mathbf{d}_3)^T[A_2] - (\mathbf{u} - \mathbf{d}_1)^T[A_1] \end{bmatrix},$$

3

$$\mathbf{c} = \begin{bmatrix} (\mathbf{d}_2^T \mathbf{d}_2 - \mathbf{d}_1^T \mathbf{d}_1) + 2(\mathbf{d}_1^T - \mathbf{d}_2^T)\mathbf{u} \\ (\mathbf{d}_3^T \mathbf{d}_3 - \mathbf{d}_1^T \mathbf{d}_1) + 2(\mathbf{d}_1^T - \mathbf{d}_3^T)\mathbf{u} \end{bmatrix}.$$

Mixed Exact-Approximate Synthesis: Computational Procedure

Problem Statement: Given the fixed pivot $\mathbf{u}$ or moving pivot $\mathbf{v}$, synthesize a planar RR dyad to guide a rigid-body from an initial position (pick) to a final position (place) while guiding it approximately through n positions.

Input : $\mathbf{p}$, $(x_i, y_i, \theta_i) \ \forall i \in \{pick, 1, \ldots, n, place\}$

Solution : The problem is solved using two approaches as follows:

Select a fixed pivot :

1. Compute $[M_i] = [A(\theta_i), \mathbf{d}_i]$ for all $i \in \{pick, 1, \ldots, n, place\}$
2. Solve Eqn. (3) using the three positions M_{pick}, M_i, M_{place} for $i = 1, \ldots, n$ to yield the set of moving pivots $\mathbf{v}_i$, $i = 1, \ldots, n$.
3. Compute $\mathbf{v} = \frac{1}{n} \sum \mathbf{v}_i$.

Select a moving pivot [12]:

1. For all $i \in \{pick, 1, \ldots, n, place\}$

    (a) Compute $[M_i] = [A(\theta_i), \mathbf{d}_i]$
    (b) Compute $\mathbf{V}_i = [A(\theta_i)]\mathbf{v} + \mathbf{d}_i$

2. Solve Eqn. (2) using the three positions M_{pick}, M_i, M_{place} for $i = 1, \ldots, n$ to yield the set of fixed pivots $\mathbf{u}_i$, $i = 1, \ldots, n$
3. Compute $\mathbf{u} = \frac{1}{n} \sum \mathbf{u}_i$.

The above algorithm is represented as a flowchart in Fig. 2.

SYNTHESIS OF OPTIMAL DYADS

The proposed constrained non-linear optimization problem maybe described as:

$$\min_{\mathbf{p}} \quad \log_e \sum_{i=1}^{n} \|cp_i - cp\|_2$$
$$\text{subject to} \quad \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} \leq \mathbf{p} \leq \begin{bmatrix} \beta_x \\ \beta_y \end{bmatrix}$$

Given the constraint space of $\mathbf{p} = [p_x, p_y]^T$, the *fmincon* tool in MATLAB is used to determine an optimal pivot $\mathbf{cp}^*$ constituting an optimal RR dyad. The optimal pivot $\mathbf{p}^*$ is then computed using the exact synthesis algorithm shown in Fig. 2 to completely determine the optimal dyad $(\mathbf{p}^*, \mathbf{cp}^*)$.
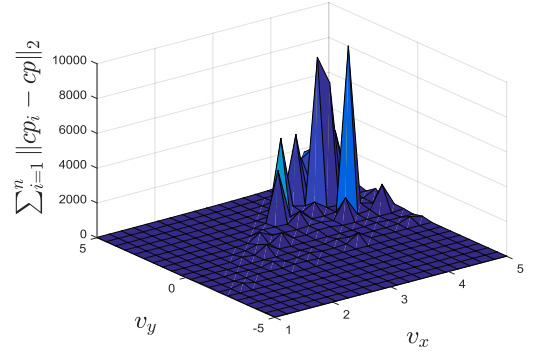
(a)
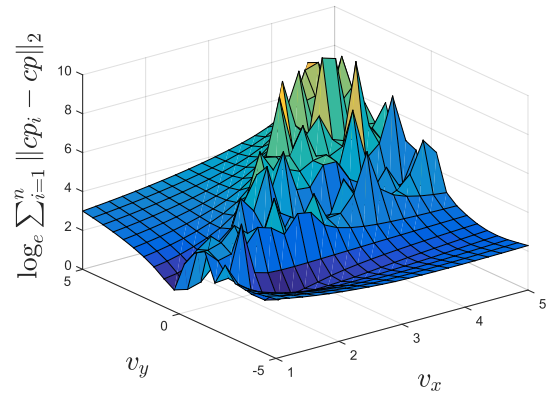
(b)

FIGURE 4. PLOT OF OBJECTIVE FUNCTION (a) UNSCALED (b) log_e SCALED

The Objective Function

The multivariate objective function $\odot_{cp}$ is a function of $\mathbf{cp}_i$, where $i = 1, \ldots, n$.

$$\odot_{cp} = \log_e \sum_{i=1}^{n} \|cp_i - cp\|_2 \qquad (4)$$

The objective function can be interpreted as an "approximation quality measure", described by Larochelle [12], which measures the degree to which the n guiding positions are being reached by each planar RR dyad. The set of $\mathbf{cp}_i$ are obtained from the n sub-problems of the three position exact synthesis for the positions $\{pick, i, place\}$, where $i = 1, \ldots, n$. The objective function calculates the deviations of the set of $\mathbf{cp}_i$ from it's mean pivot $\mathbf{cp}$ using the Euclidean distance between each $\mathbf{cp}_i$ and $\mathbf{cp}$. This distance is scaled using $\log_e$ which 'scales up' the set of $\mathbf{cp}_i$ closer to it's mean $\mathbf{cp}$ and the 'scales down' the set of $\mathbf{cp}_i$ farther from it's mean $\mathbf{cp}$. The $\log_e$ scaling brings out peak, valley and ridge
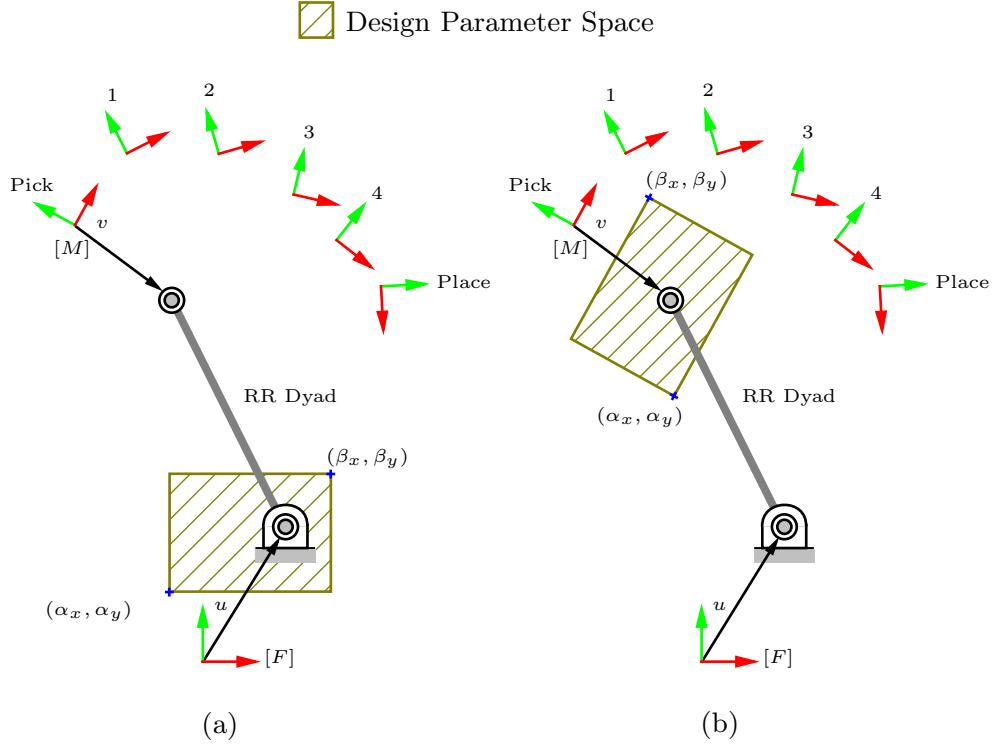
FIGURE 3. DESIGN PARAMETER SPACE (a) FIXED PIVOT SPACE (b) MOVING PIVOT SPACE

characteristics of the objective function as shown in Fig. 4(b) and comparing it with an unscaled version of the same objective function in Fig. 4(a). Note that $\mathbf{cp} = \mathbf{u}$ in Fig. 4. The $\log_e$ scaling helps in finding better local minima solutions.

Design Parameter Space

The choice of $\mathbf{p} = [p_x, p_y]^T$ is a design parameter whose space is defined by the inequality constraints of the optimization problem. The parameters $(\alpha_x, \beta_x, \alpha_y, \beta_y)$ bound this design parameter space such that,

$$\begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} \leq \mathbf{p} \leq \begin{bmatrix} \beta_x \\ \beta_y \end{bmatrix} \tag{5}$$

The fixed pivot space is defined with respect to $[F]$ as shown in Fig. 3(a) and the moving pivot space is defined with respect to $[M]$ as shown in Fig. 3(b).

Discretization of Design Parameter Space

In order to determine the global minimum within the bounded design parameter space, a multi-start approach is utilized. The design parameter space is discretized with a resolu-

tion determined by the designer and stored in $[R]$. Each discrete element of $[R]$ will serve as a starting point for the optimization routine, refer Fig. 2.

Synthesis of Optimal Dyad: Computational Procedure

Problem Statement: Synthesize an optimal planar RR dyad $(\mathbf{p}^*, \mathbf{cp}^*)$ to guide a rigid-body from an initial position (pick) to a final position (place) while guiding it approximately through n positions.

Input: $\mathbf{p}$, $(x_i, y_i, \theta_i) \ \forall i \in \{pick, 1, \dots, n, place\}$, design parameter space constraints of $\mathbf{p} = [p_x, p_y]^T$ such that $\alpha_x \leq p_x \leq \beta_x$, $\alpha_y \leq p_y \leq \beta_y$.

Solution:

1. Compute $[M_i] = [A(\theta_i), \mathbf{d}_i]$ for all $i \in \{pick, 1, \dots, n, place\}$
2. Discretize the design space constraint and store it in $[R]$. Each element of $[R]$ will serve as a starting point
3. For each element of $\mathbf{p} \in [R]$,

   (a) Compute $\mathbf{cp}$ using exact synthesis algorithm for M_{pick}, M_i, M_{place}
   (b) Minimize $\odot_{cp} = \log_e \sum_{i=1}^n \|\mathbf{cp}_i - \mathbf{cp}\|_2$

4. Find $\mathbf{cp}^*$ for the least value of $\odot_{cp}$
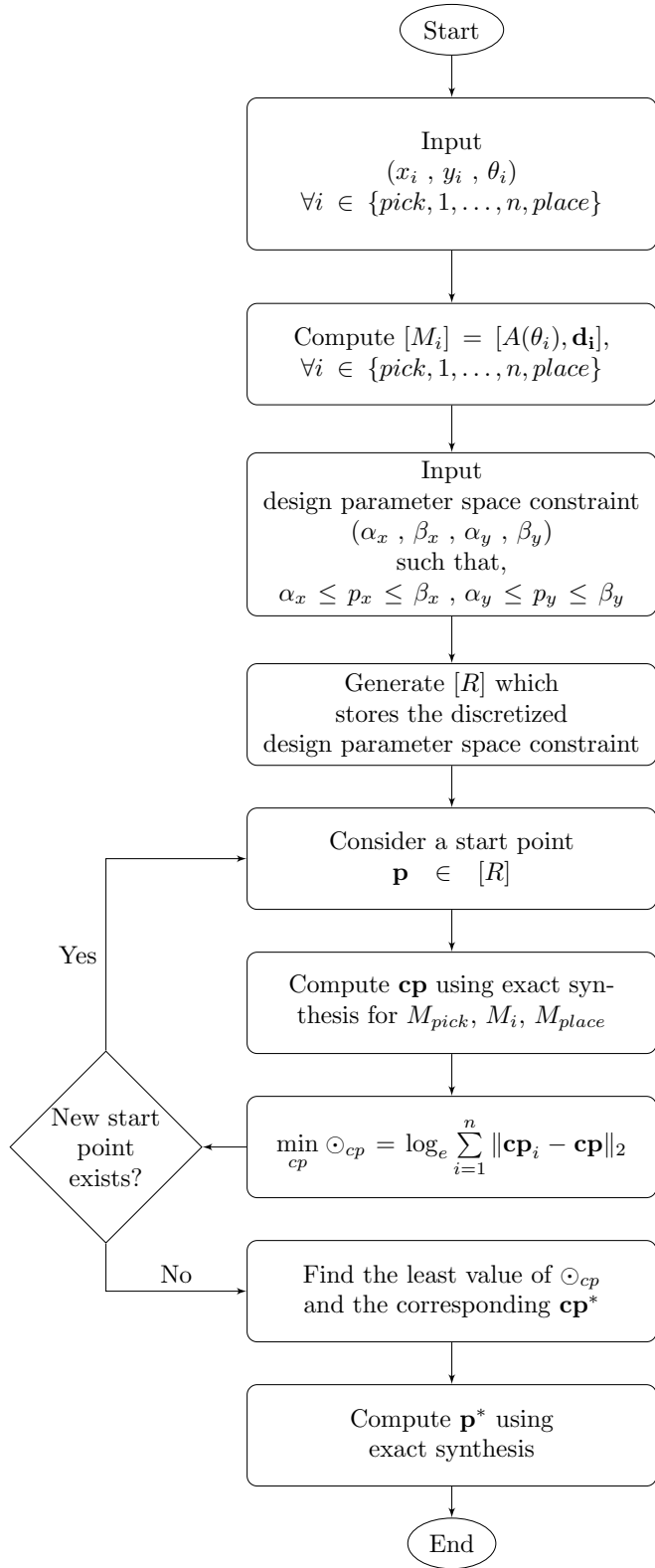5. Compute $\mathbf{p}^*$ using exact synthesis

FIGURE 5. ALGORITHM TO SYNTHESIZE OPTIMAL RR DYADS

The above algorithm is described as a flowchart in Fig. 5. Since, two dyads may be combined to form a four-bar mechanism, the algorithm is executed twice; once for each dyad. The optimal dyads are denoted as, dyad a: $(\mathbf{p}_a^*, \mathbf{cp}_a^*)$ and dyad b: $(\mathbf{p}_b^*, \mathbf{cp}_b^*)$.

EXAMPLE: 11 POSITION PROBLEM

The proposed algorithm is now employed to synthesize a planar four-bar mechanism for 11 positions; with pick and place positions along with nine guiding positions as shown in Table 1. This rigid-body guidance problem was proposed by McCarthy at the 2002 ASME Conference [11]. The example was run on a 3.20 GHz Intel® i5 Core™ CPU. The computation time for each case study took about 100 seconds on an average for each dyad when the optimization routine used 900 starting points. The mixed exact-approximate synthesis algorithm allows the designer to synthesize an RR dyad by either choosing the fixed pivot **u** or moving pivot **v** as design parameter space constraints. An optimal four-bar mechanism is obtained by combining two optimal RR dyads a: $(\mathbf{p}_a^*, \mathbf{cp}_a^*)$ and b: $(\mathbf{p}_b^*, \mathbf{cp}_b^*)$ which may be synthesized by the following two ways:

Case 1: Constrained $\mathbf{u}_a$ and $\mathbf{u}_b$

The following design constraint space in $\mathbf{u}_a = [u_{a_x}, u_{a_y}]^T$ and $\mathbf{u}_b = [u_{b_x}, u_{b_y}]^T$ is considered:

$$0 \le u_{a_x} \le 5$$
$$0 \le u_{a_y} \le 2$$
$$-5 \le u_{b_x} \le 1$$
$$-5 \le u_{b_y} \le 1$$

The synthesis of optimal dyads algorithm described in Fig. 5 yielded optimal dyad a: $\mathbf{u}_a^* = [2.1991, 1.6465]^T$, $\mathbf{v}_a^* = [1.4245, -1.9397]^T$ with an objective function score of $\odot_{v_a} = 0.1522$, and optimal dyad b: $\mathbf{u}_b^* = [0.8008, 0.3536]^T$, $\mathbf{v}_b^* = [1.5754, -0.0602]^T$ with an objective function score of $\odot_{v_b} = 0.1523$. Based on these optimal pivots, a $0 - \pi$ double-rocker four-bar mechanism was obtained, as shown in Fig. 6.

Case 2: Constrained $\mathbf{v}_a$ and $\mathbf{v}_b$

The following design constraint space in $\mathbf{v}_a = [v_{a_x}, v_{a_y}]^T$ and $\mathbf{v}_b = [v_{b_x}, v_{b_y}]^T$ is considered:

$$0 \le v_{a_x} \le 4$$
$$0 \le v_{a_y} \le 20$$
$$1 \le v_{b_x} \le 5$$
$$-5 \le v_{b_y} \le 5$$

TABLE 1.   ELEVEN POSITION PROBLEM

#	x	y	θ (deg)	Motion Type
1	-1	-1	90	exact
2	-1.2390	-0.5529	77.3621	approximate
3	-1.4204	0.3232	55.0347	approximate
4	-1.1668	1.2858	30.1974	approximate
5	-0.5657	1.8871	10.0210	approximate
6	-0.0292	1.9547	1.7120	approximate
7	0.2632	1.5598	10.0300	approximate
8	0.5679	0.9339	30.1974	approximate
9	1.0621	0.3645	55.0346	approximate
10	1.6311	0.0632	77.3620	approximate
11	2	0	90	exact

The synthesis of optimal dyads algorithm described in Fig. 5 yielded optimal dyad a: $\mathbf{u}_a^* = [0.6744, 0.4559]^T$, $\mathbf{v}_a^* = [1.4985, 0.0064]^T$ with an objective function score of $\odot_{v_a} = 0.4979$, and optimal dyad b: $\mathbf{u}_b^* = [0.7934, 0.3758]^T$, $\mathbf{v}_b^* = [1.5976, -0.0528]^T$ with an objective function score of $\odot_{u_b} = 0.9522$. Based on these optimal pivots, a $\pi - \pi$ double-rocker four-bar mechanism was obtained, as shown in Fig. 7.

Interpretation

The above case study was simulated in MATLAB and a screenshot of each optimal four-bar linkage is represented in Fig. 6 and Fig. 7. By visually inspecting the simulation, it was clear that the optimal linkage represented in Fig. 6 seems to approximately reach all 11-positions whereas the same could not be stated for the optimal linkage represented in Fig. 7. A driving RR dyad [13] may be added to this optimal four-bar linkage to form a Watt-II linkage and also to limit the motion of the optimal four-bar linkage.

CONCLUSION

The synthesis of RR dyads for *n* position guidance problems using mixed exact-approximate synthesis has been performed. MATLAB's constrained non-linear optimization tool has been used to run the optimization algorithm that computed optimal pivots within a design parameter constraint space. The proposed algorithm has been tested against McCarthy's 11-position rigid-body guidance problem stated at the 2002 ASME Conference [11].

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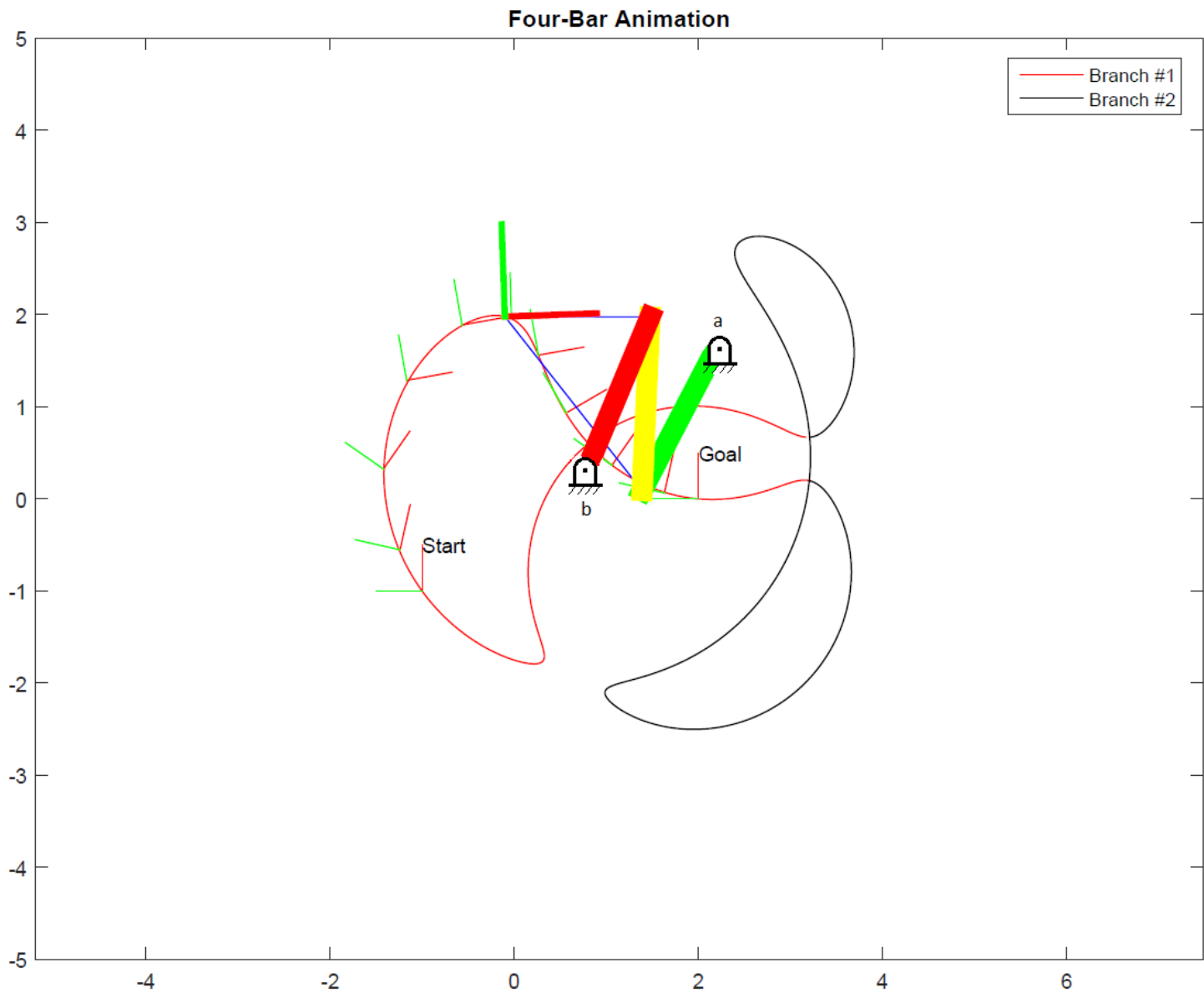
FIGURE 6. CASE 1: OPTIMAL 4-BAR LINKAGE

[13] Norton, R. L., ed., 2013. *Design of Machinery: An Introduction to the Synthesis and Analysis of Mechanisms and Machines*, 5th ed. Prentice Hall, New York, NY.
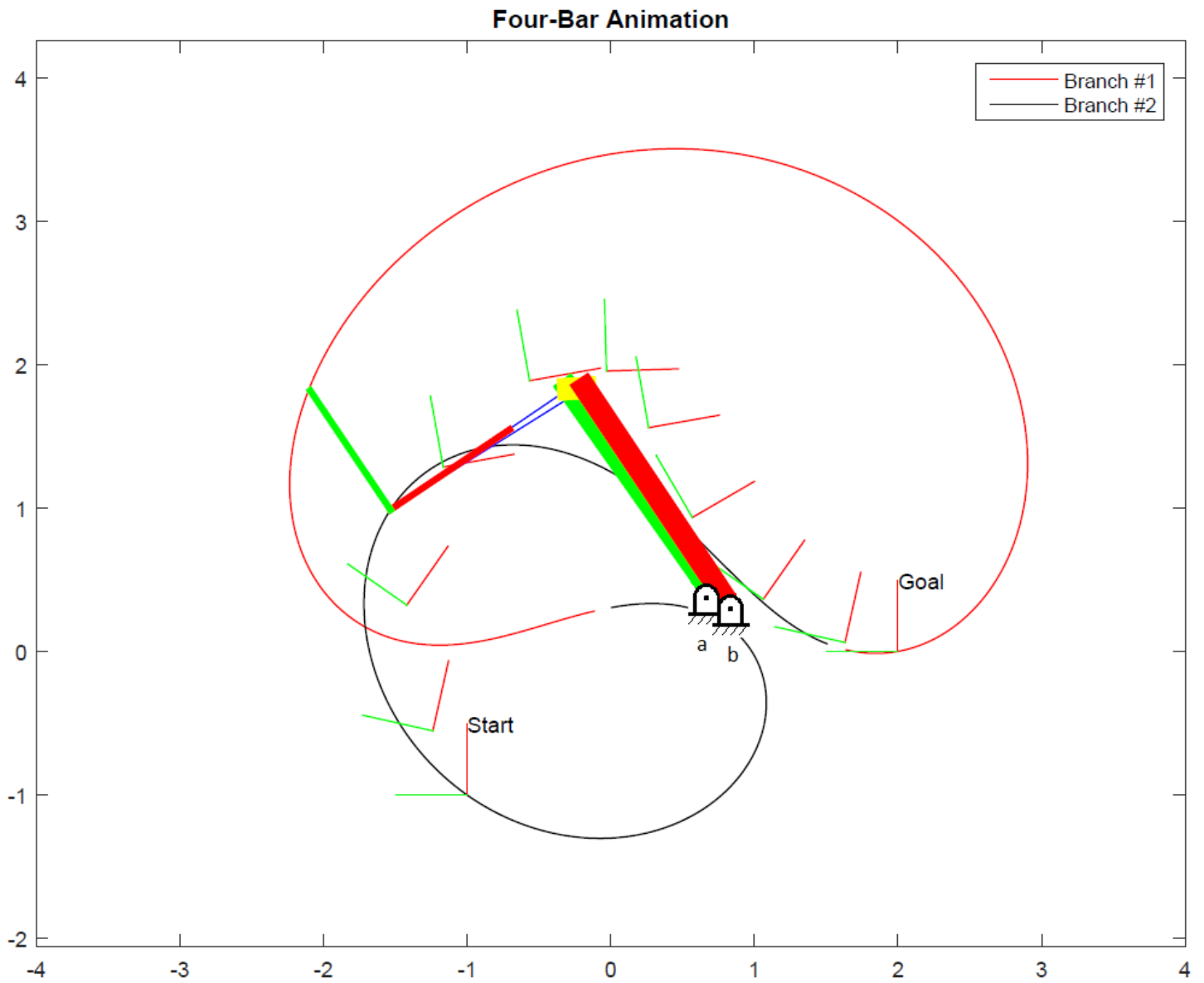
Four-Bar Animation

FIGURE 7.   CASE 2: OPTIMAL 4-BAR LINKAGE

9